Existence of Optimal Quasi-Metrics and Consequences for Hardy Spaces

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Abstract: For a quasi-metric ρ on a set X, the optimal constant C which makes true the quasi-ultrametric condition

$$\rho(x,y) \leq C \max\{\rho(x,z), \rho(z,y)\}, \qquad \text{for all } x,y,z \in X,$$

determines, among other things, precisely how much Hölder regularity the function ρ exhibits. A natural question is when given a quasi-metric ρ, if one examines all quasi-metrics which are pointwise equivalent to ρ, does there exist one which exhibits an optimal amount of Hölder regularity (or, equivalently, is most like an ultrametric)? We show that, under certain constraints, the answer is affirmative, and further demonstrate that the answer, in general, is negative, proven by constructing a suitable Rolewicz-Orlicz space. Lastly, we state what ramifications this has regarding the amount of analysis which may be carried out on certain Hardy spaces.