ADE Dynkin diagrams in algebra, geometry and beyond – based on work of Ellen Kirkman –

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Happy Birthday, Ellen

A Theorem of Lorenz-Lorenz (1995):

Let H be a Hopf algebra. Then

$$\mathbf{gldim}\, H = \mathbf{projdim}\, (H / \ker \epsilon).$$

This is a very useful result, and an example of

“easy to state”

and

“extremely powerful”

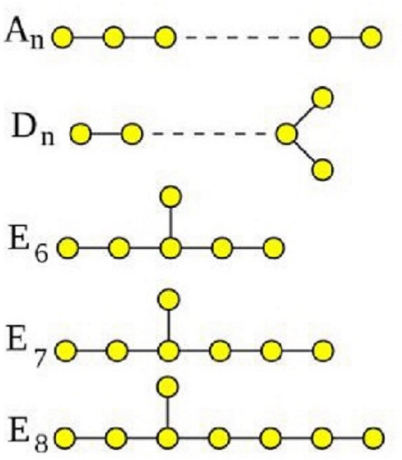
mathematics.

Happy Birthday, Martin

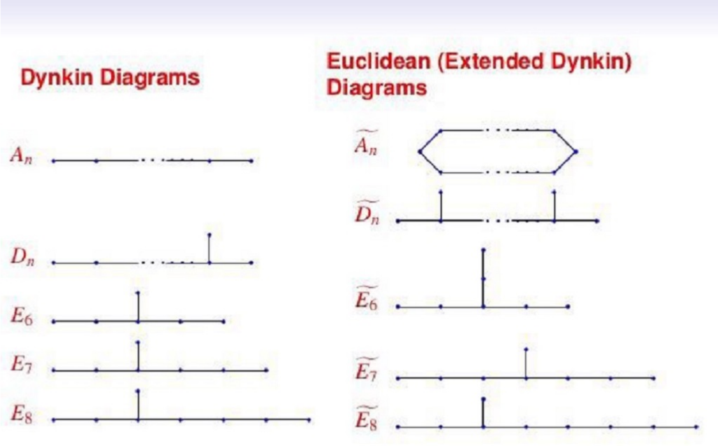
Contents

What are the ADE diagrams/graphs?

Recall from Georgia's talk:



ADE vs $\widetilde{A}\widetilde{D}\widetilde{E}$



ADE quivers play an important role in

♣ representation theory of finite dimensional algebras and quivers

(♣1) Theorem (Gabriel 1972): Let Q be a finite connected quiver. The path algebra $\mathbb{C}Q$ is of finite representation type if and only if the underlying graph of Q is a Dynkin graph of type ADE.

(♣2) Theorem (Nazarova and Donovan-Freislich 1973): Let Q be a finite connected quiver. The path algebra $\mathbb{C}Q$ is of tame representation type if and only if the underlying graph of Q is an extended Dynkin graph of type $\widetilde{A}\widetilde{D}\widetilde{E}$.

(♣3) Theorem (Drozd's Trichotomy Theorem 1980): Every finite dimensional algebra is either of

- (•) finite, or
- (•) tame, or
- (•) wild representation type.

Example of Drozd's Trichotomy Theorem

Finite representation type (A D E)

$$A_3: \quad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

Tame type $(\widetilde{A}\,\widetilde{D}\,\widetilde{E})$

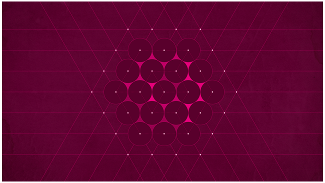
$$\widetilde{D}_5: \quad \begin{tikzcd}
1 \arrow[dr] & & & 0 \arrow[dl] \\
& 3 \arrow[r] & 4 & \\
2 \arrow[ur] & & & 5 \arrow[ul]
\end{tikzcd}$$

Wild type (Not A D E/ $\widetilde{A}\,\widetilde{D}\,\widetilde{E}$)

$$K_2(3): \quad \begin{tikzcd}
\underset{1}{\bullet} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \underset{2}{\bullet}
\end{tikzcd}$$

ADE root systems are related to

2-dim Sphere Packing solution: honeycomb



3-dim Sphere Packing solution: Orange stacking

ADE root systems are related to

♢ Sphere Packing Problem (started in 1611 by Johannes Kepler)

(♢1) $d = 1$: trivial. $d = 2$: Thue 1892 and Fejes Toth 1940.

(♢2) $d = 3$ called Kepler conjecture: Hales 1998 (announced), Ann. of Math. 2005.

Hales and Ferguson received the Fulkerson Prize for outstanding papers in the area of discrete mathematics for 2009.

(♢3) Solutions to the sphere packing problem in dimension 8 by Viazovska by using E_8, – E_8 is the best – (Preprint, March 14, 2016; Ann. of Math. 2017)

(♢4) dimension 24 by Cohn-Kumar-Miller-Radchenko-Viazovska (March 21, 2016, a week late; Ann. of Math. 2017) .

(♢5) Still open for any dim, except for $1, 2, 3, 8, 24$.

ADE graphs are used in

♡ Physics and geometry:

$\downarrow$

(super)conformal field theories ((S)CFTs 2d or 6d), and more generally quantum field theories (QFTs in different dimensions), quantum or supersymmetric gauge theories (QGTs), asymptotic locally Euclidean spaces (ALE spaces) etc.

♠ Combinatorics and linear algebra:

$\downarrow$

classification of simple graphs by spectral radius, or matrices over $\mathbb{N}$ with norm less than 2.

$\downarrow$

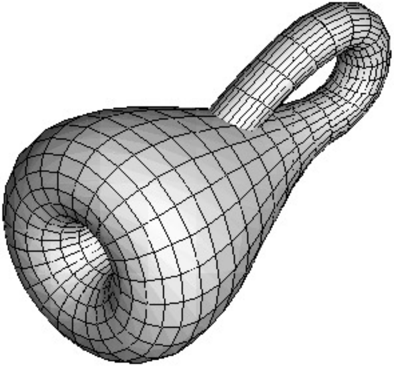
♢ Cluster algebras (Milen's talk), and

$\downarrow$

♣ Many others (Simon's talk, Bob's talk, ...)

Felix Klein's work

The most famous: Klein bottle



Klein's story

Klein classified all finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$ (or of $\mathrm{SU}_2(\mathbb{C})$) in 1890's

This classification is closely connected with

$$\downarrow$$

♠ Platonic Solids known to the ancient Greeks around 360 B.C.

♣ simple surface singularities (also called rational double points, du Val singularities, Kleinian singularities). (Details were given in Simon's talk.)

♢ classification of simple Lie algebras (root systems and ADE Dynkin graphs).

♡ Coxeter groups, reflection groups, and finite simple groups.


Pythagorean Solids
(The Platonic Solids)
TETRA HEDRON (FOUR-SIDED)
HEXA HEDRON (SIX-SIDED)
OCTA HEDRON (EIGHT-SIDED)
ICOSA HEDRON (TWENTY-SIDED)
DODECA HEDRON (TWELVE-SIDED)
FIRE
EARTH
AIR
WATER
AETHER

Continuation of Klein's story (and continuation of Georgia's , Simon's ... talk)

In 1980, John McKay observed that there is a one-to-one correspondence between **finite subgroups** G in Klein's classification and **extended Dynkin graphs** (i.e. $\widetilde{A}\widetilde{D}\widetilde{E}$ types):

$$\text{finite subgroups } \; G \subseteq \mathrm{SL}_2(\mathbb{C}) \quad \longleftrightarrow \quad \widetilde{A}\widetilde{D}\widetilde{E} \; \text{ graphs.}$$

The Dynkin graphs associated to these G are called the *McKay quiver* of G, "which can be obtained by using the fusion ring of the irreducible representations of G". (Definition was given in Georgia's talk.)

Example 1: (From Georgia's talk) $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a^n = 1 \right\} \subseteq SL_2(\mathbb{C})$. There are n simple G-modules $S_0, S_1, \cdots S_{n-1}$ (these are vertices of $\widetilde{A}_{n-1}$). The embedding of $G \subseteq SL_2(\mathbb{C})$ determines a 2-dimensional G-module V. Then the fusion rule is given by

$$S_i \otimes V \cong S_{i-1} \oplus S_{i+1}$$

where the index i is in $\mathbb{Z}/(n)$. The fusion rule gives rise to the arrows from

$$S_{i-1} \longleftarrow S_i \longrightarrow S_{i+1}.$$

This is how we get the extended Dynkin quiver $\widetilde{A}_{n-1}$.

$\widetilde{A}\widetilde{D}\widetilde{E}$

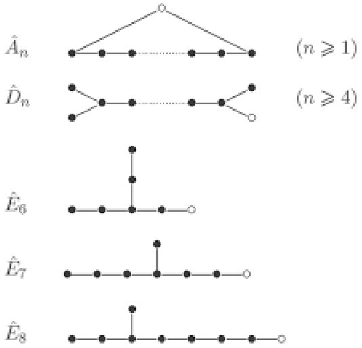
$$
\begin{tikzcd}
 & S_1 \arrow[r, shift left] \arrow[dl, shift left] & \cdots \arrow[r, shift left] \arrow[l, shift left] & S_{i-1} \arrow[dr, shift left] \arrow[l, shift left] & \\
S_0 \arrow[ur, shift left] \arrow[dr, shift left] & & & & S_i \arrow[ul, shift left, red] \arrow[dl, shift left, red] \\
 & S_{n-1} \arrow[r, shift left] \arrow[ul, shift left] & \cdots \arrow[r, shift left] \arrow[l, shift left] & S_{i+1} \arrow[ur, shift left] \arrow[l, shift left] &
\end{tikzcd}
$$

McKay quiver $\widetilde{A}_{n-1}$

It is amazing that

every finite subgroup $G \subseteq SL_2(\mathbb{C})$ is determined by a simple quiver of type $\widetilde{A}\widetilde{D}\widetilde{E}$.

$\widetilde{A}\widetilde{D}\widetilde{E}$ graphs again



$\longrightarrow$ $\widetilde{A}\widetilde{D}\widetilde{E}$ is obtained by adding "$\circ$" to ADE $\longleftarrow$

Geometric McKay Correspondence

McKay also observed that the desingularization graph of the Kleinian singularities $\mathbb{C}^2/G$ is again the McKay quiver of type ADE.

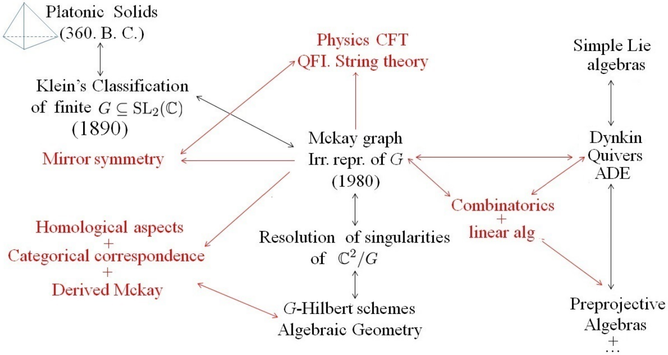
Artin and Verdier in 1985 worked out a geometric McKay correspondence between indecomposable **non-free MCM** $\mathbb{C}[x,y]^G$-modules and **irreducible components of the exceptional fiber** of the resolution of $\mathbb{C}^2/G$.

$$MCM \Longleftrightarrow \text{components of the exceptional fiber} \Longleftrightarrow ADE$$

$\Longrightarrow$ Related to the study of G-Hilbert schemes.
$\Longrightarrow$ Extensions to higher dimensions.

(McKay correspondence was also mentioned in Andrew's talk.)

Web



Platonic Solids
(360. B. C.)
Klein's Classification
of finite $G \subseteq SL_2(\mathbb{C})$
(1890)
Physics CFT
QFI. String theory
Simple Lie
algebras
Mirror symmetry
Mckay graph
Irr. repr. of G
(1980)
Dynkin
Quivers
ADE
Combinatorics
+
linear alg
Homological aspects
+
Categorical correspondence
+
Derived Mckay
Resolution of singularities
of $\mathbb{C}^2/G$
G-Hilbert schemes
Algebraic Geometry
Preprojective
Algebras
+
...

New developments during the last 30 years

Next we will focus on the following three subjects

A： Quantum Group

+

Hopf Algebra

B： NC Invariant Theory

（NC Algebra）

C： NC Algebraic

Geometry

A: Quantum Group (and Hopf Algebra)

Let H be a semisimple Hopf algebra and V a representation of H. We define a McKay functor

$$\mathcal{M} : \{(H,V)\} \to \{quivers\}.$$

Following Georgia's talk, the McKay quiver $\mathcal{M}(H,V)$ is defined by

- vertices: iso-classes of irred. repr. of H, say $I_1, I_2, \cdots, I_n$.
- arrows: $I_i \xrightarrow{n_{ij}} I_j$ if

$$I_i \otimes V = \sum_{j=1}^{n} I_j^{\oplus n_{ij}}.$$

$\longrightarrow$ Understanding all quivers $\mathcal{M}(H,V)$ is a very interesting and difficult project. $\longleftarrow$

A: Quantum Group (and Hopf Algebra)

Theorem 2 (Happel-Preiser-Ringel 1980, Chan-Kirkman-Walton 2016): If the H-representation V is a 2-dimensional, self-dual, inner-faithful, then the underlying graph $\underline{\mathcal{M}(H,V)}$ is of type $\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1$.

$\widetilde{DL}_n$ graph (given in Simon's talk)

*** $\widetilde{L}_1$ was also given in Simon's talk. ***

$$\{some\ \ (H,V)s\} \xrightarrow{\mathcal{M}} \{\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1\}.$$

A: Quantum Group (and Hopf Algebra)

Remark 3: For each $Q \in \{\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1\}$, there are at most 3 non-isom H such that $\underline{\mathcal{M}(H,V)} = Q$.

$$\{(H,V)\} \xrightarrow{3-1} \{\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1\}.$$

Remark 4: When restricted Theorem to group algebras $H = \mathbb{C}G$ for finite groups $G \subseteq SL_2(\mathbb{C})$, there is a 1-1 (McKay) Correspondence

$$\{(\mathbb{C}G,V)\} \xrightarrow{1-1} \{\widetilde{A}\widetilde{D}\widetilde{E}\}.$$

Skew $\widetilde{A}\widetilde{D}\widetilde{E}$ – a technical generalization

The *spectral radius* of a matrix M, denoted by $\rho(M)$, is defined to be the maximum absolute value of the eigenvalues of M.
The *spectral radius* of a graph (or a quiver) G, denoted by $\rho(G)$, is defined to be $\rho(M)$, where M is the adjacency matrix of G.
Recall that a graph is called *simple* if it has no loop or double edges. This means that, considered G as a quiver, for each arrow from i to j, there is a unique arrow from j to i for every pair of vertices i and j.
A quiver Q is called a *skew simple graph* if there is an automorphism σ of Q such that, for each arrow from i to j, there is a unique arrow from j to $\sigma(i)$ for every pair of vertices i and j.
A result of Smith states that a connected simple graph G has $\rho(G) = 2$ if and only if G is of type $\widetilde{A}\widetilde{D}\widetilde{E}$.
Motivated by the above result, we say a quiver Q is of *skew* $\widetilde{A}\widetilde{D}\widetilde{E}$ if Q is a skew simple graph and $\rho(Q) = 2$.

B: NC Invariant Theory and NC Algebra

Suppose now B is a (connected graded) algebra

$$B = \mathbb{C} \oplus B_1 \oplus B_2 \oplus \cdots$$

with Auslander-Reiten (AR) sequences.
We define Auslander-Reiten quiver (or AR-quiver)

$$\mathcal{A} : \{B\} \longrightarrow \{quivers\}$$

as follows.
The Auslander-Reiten quiver $\mathcal{A}(B)$ is defined by
• vertices: iso-classes of indecomposable graded maximal Cohen-Macaulay (MCM) B-modules, say $C_1, \cdots, C_n$.
• arrows: $C_i \xrightarrow{n_{ij}} C_j$ if n_{ij} is the multiplicity of C_i in the middle term of the AR-sequence ending with C_j.

B: NC Invariant Theory and NC Algebra

Understanding the AR-quivers is one of most important project in representation theory of finite dimensional algebras, commutative local algebras, and now in NC algebra.

Theorem 5 (Chan-Kirkman-Walton 2016): Let B be a noetherian CM graded algebra of GKdim 2 that is of finite CM type. Suppose that B is module-finite over a central affine graded subalgebra. Then $\mathcal{A}(B)$ is of type $skew\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1$.

We have

$$\{some \;\; B\} \xrightarrow{\mathcal{A}} \{skew\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1\}.$$

Remark 6: The map above is

$$\{B\} \xrightarrow{\quad \infty - 1 \quad} \{skew\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1\}.$$

B: NC Invariant Theory and NC Algebra

Say B is PI if it is module finite over a central commutative subring.

Conjecture 7: The "PI" hypothesis should be removed.

Question 8: Why is this related to NC invariant theory?

Example 9: If R is noetherian PI AS regular of global dimension 2, and H a semisimple Hopf algebra acting on R with trivial homological determinant, then the invariant subring $B := R^H$ satisfies hypotheses of the above Theorem.

Most known examples are related to the group actions and Hopf algebra actions (or quotient singularities in geometry).

C: NC Algebraic geometry

NC projective geometry was initiated by Artin, Schelter, Tate and Van den Bergh (ASTV) in 1980s.
First we consider $\mathbb{N}$-graded basic algebras:

$$A = \mathbb{C}^{\oplus m} \oplus A_1 \oplus A_2 \oplus \cdots$$

satisfying
- $\dim_{\mathbb{C}} A_i < \infty$,
- A is noetherian.

If $m = 1$, then A is called connected graded.

Let $S_1, \cdots, S_m$ be 1-dimensional simple A-modules.

C: NC Algebraic geometry

Definition 10: (From Dan's talk) A noetherian $\mathbb{N}$-graded basic algebra R is called *Artin-Schelter regular* (or AS regular for short) if

(i) $\operatorname{gldim} R = d < \infty$,

(ii) $\dim_{\mathbb{C}} \operatorname{Ext}^i_R(S_i, R) = \begin{cases} 0 & i \neq d, \\ 1 & i = d \end{cases}$ for all 1-dimensional graded modules S_i. (Here S_i is the 1-dimensional simple A-modules.)

A partial list of important classification results:

(1) ASTV (1980s-1990s) classified connected graded AS regular algebras of global dimension 3, which also classified NC $\mathbb{P}^2$s.

(2) Bocklandt (2006) classified basic AS regular algebras of global dimension 2 that are Calabi-Yau.

(3) Reyes-Rogalski (2017) classified $\mathbb{N}$-graded (non-connected) AS regular algebras of global dimension 2.

C: NC Algebraic geometry

Study of AS regular algebras is one of main projects in NC Algebraic Geometry.

We can define Gabriel functor

$$\{basic\ AS\ regular\ algebras\} \xrightarrow{\mathcal{G}} \{quivers\}.$$

The Gabriel quiver $\mathcal{G}(R)$ is defined by
- vertices: 1-dimensional graded simples $S_1, \cdots, S_m$.
- arrows: $S_i \xrightarrow{n_{ij}} S_j$ if $n_{ij} = \dim_{\mathbb{C}} \operatorname{Ext}^1_R(S_i, S_j)$.

It is very interesting to understand all Gabriel quivers $\mathcal{G}(R)$.

C: NC Algebraic geometry

Theorem 11 (Reyes-Rogalski 2017, Qin-Wang (2018?)): If R is AS regular of global dimension 2, then $\mathcal{G}(R)$ is of type $skew\widetilde{A}\widetilde{D}\widetilde{E}/\widetilde{DL}/\widetilde{L}_1$.

Note that:

Reiten-Van den bergh 1989: for graded tame orders.

Bocklandt 2006: for Calabi-Yau R, $\mathcal{G}(R)$ is of type $\widetilde{A}\widetilde{D}\widetilde{E}$.

Chen-Kirkman-Walton 2016: for smash products with trivial homological determinant.

Reyes-Rogalski 2017: most general, ($\mathbb{N}$-graded).

Qin-Wang 2018?: work in progress, for noncommutative resolutions (commonly $\mathbb{Z}$-graded).

Next we have quizzes.

All three subjects are related to the ADE graphs: Q1

A： Quantum Group

+

Hopf Algebra

A D E / D L / L

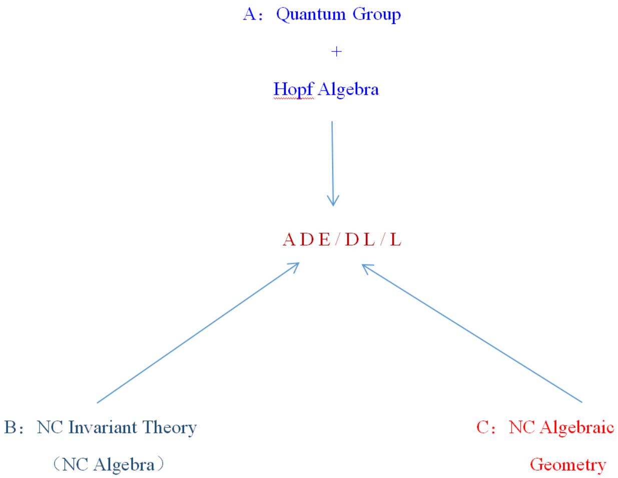
B： NC Invariant Theory

（NC Algebra）

C： NC Algebraic

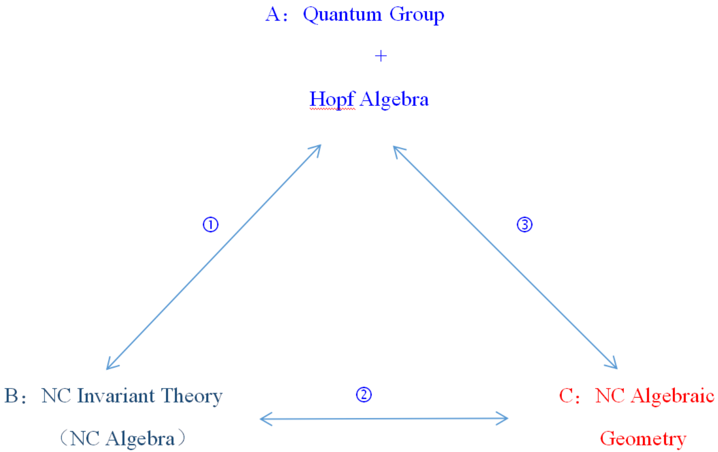
Geometry

All three subjects are related to the ADE graphs: D_4



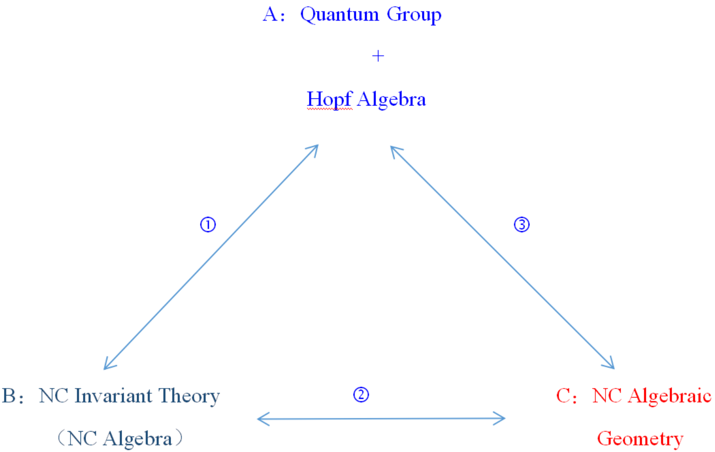
Direct connections: Q2

We would like to see some direct connections:



Direct connections: $\widetilde{A}_2$

We would like to see some direct connections:



Connection 1

<u>Theorem</u> 12 (Chan-Kirkman-Walton 2016): Given (H, V) as in **Theorem** 2, one can construct an algebra B as in **Theorem** 5 such that the **McKay quiver** $\mathcal{M}(H, V)$ = the **Auslander-Reiten** quiver $\mathcal{A}(B)$.

$\longrightarrow$ B is not unique. $\longleftarrow$

<u>A sketch proof</u>: Given (H, V), we can construct a connected graded regular algebra R of global dimension 2, called R, such that H acts on R with trivial homological determinant and $R_1 = V$. Then we take B to be the fixed subring R^H. By a result of Chen-Kirkman-Walton, $\mathcal{M}(H, V) = \mathcal{A}(B)$.

The definitions of two quivers $\mathcal{M}(H, V)$ and $\mathcal{A}(B)$ are very different, but we find a way to connect these two quivers.

Connection 2

Using Van den bergh's NC crepant resolutions, we have

Theorem 13 (Chan-Kirkman-Walton 2016): Let B be as in **Theorem** 5, one can construct a basic AS regular algebra R such that R is Morita equivalent to $\mathrm{End}_B(\bigoplus MCMs)$ and the **Auslander-Reiten** quiver $\mathcal{A}(B)$ = the **Gabriel quiver** $\mathcal{G}(R)$.

$\longrightarrow$ In this case (in Theorem 5) R is PI. $\longleftarrow$

When B is commutative, this is a result of Leuschke (2007).

We conjecture that Theorem 13 holds for non-PI algebras.

Connection 3

Theorem 14 (Chan-Kirkman-Walton 2016): Given (H, V) as in **Theorem** 2, one can construct a connected graded AS-regular algebra H and an H-action on R such that $R\#H$ is Morita equivalent to a basic AS regular algebra R, and the **McKay quiver** $\mathcal{M}(H, V) =$ the **Gabriel quiver** $\mathcal{G}(R)$.

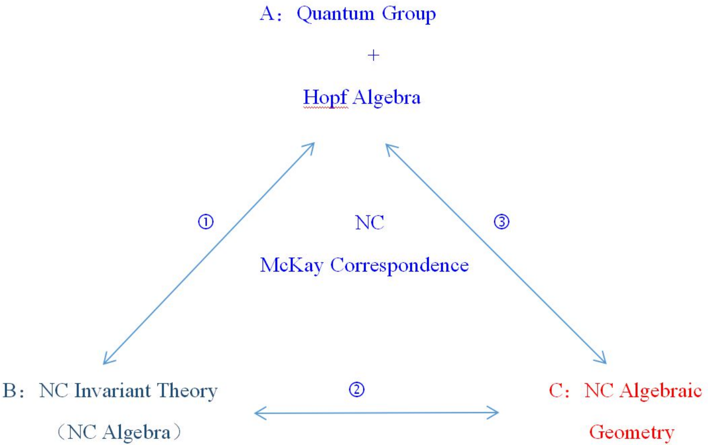
Note that in this case we do not assume that R is PI.

This theorem relates

- (H, V),
- H-action on R,
- the fixed subring R^H, and
- the smash product algebra $R\#H$.

NC McKay Correspondence

The project involving all these subjects and these connections together is called *NC McKay Correspondence*.



Our goal

One of our research goals is to search for

simple,

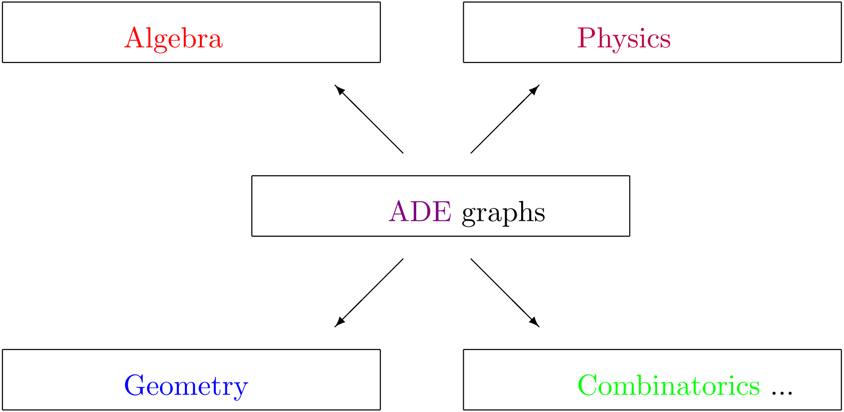
elegant,

deep, and

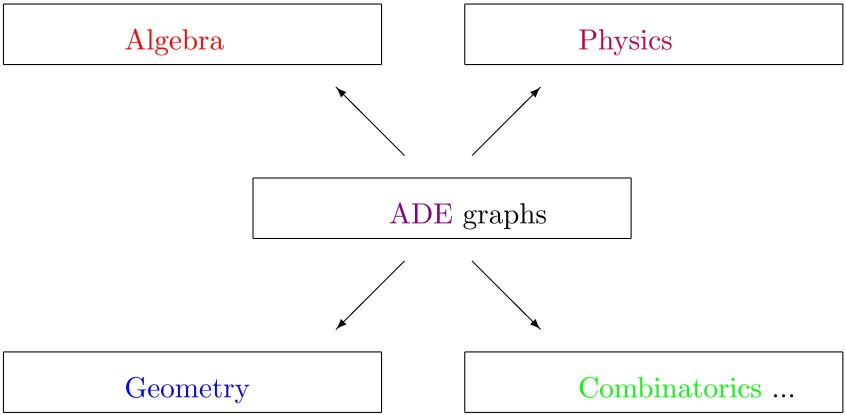
fundamental

connections between different areas of Mathematics.

ADE is such an example: Q3



ADE is such an example: $\widetilde{D}_4$



Dear Ellen and Martin,