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HYPERBOLIC CONVEXITY OF A STANDARD FUNDAMENTAL DOMAIN OF A SUBGROUP OF A HECKE DISCRETE GROUP

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Omer Yayenie
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ABSTRACT

HYPERBOLIC CONVEXITY OF A STANDARD FUNDAMENTAL DOMAIN OF A SUBGROUP OF A HECKE DISCRETE GROUP

Omer Yayenie

DOCTOR OF PHILOSOPHY

Temple University, August, 2003

Professor Marvin I.Knopp, Chair

In this thesis we study the most important and practical method of obtaining a fundamental domain through the use of a right coset decomposition as described below. The sets

$$\mathcal{R}_0 = \left\{\tau \in \mathbb{H} : |\tau| > 1 \ \ \& \ \ |Re(\tau)| < \frac{1}{2}\right\} \tag{0.1}$$

and

$$\mathcal{R}^0 = \left\{\tau \in \mathbb{H} : |\tau - 1| > 1 \ \ \& \ \ 0 < Re(\tau) < \frac{1}{2}\right\}, \tag{0.2}$$

are fundamental domains for $\Gamma(1)$. These sets are hyperbolically convex and simply connected domains in the upper half-plane. It is well known that if Γ is s subgroup of $\Gamma(1)$, and

$$\Gamma(1) = \Gamma \cdot \{A_1, A_2, \cdots, A_\mu\}, \tag{0.3}$$

then the set

$$\mathcal{R}_\Gamma = \left(\overline{\bigcup_{k=1}^{\mu} A_k(\mathcal{R}_0)}\right)^o \tag{0.4}$$

is a fundamental domain, called *a standard fundamental domain*, for Γ. It is also known that if $[\Gamma(1) : \Gamma] = \mu < \infty$, then there exists a finite number of elements $C_1, \cdots, C_p \in \Gamma(1)$ and $\lambda_1, \cdots, \lambda_p \in \mathbb{N}$ such that

$$\Gamma(1) = \Gamma \cdot \bigcup_{k=1}^{p} \{C_k, C_kS, ..., C_kS^{\lambda_k - 1}\}. \tag{0.5}$$

Applying the above result we find that the set

$$\mathcal{R}_\Gamma = \left(\overline{\bigcup_{k=1}^{p} \bigcup_{j=1}^{\lambda_k - 1} C_k S^j(\mathcal{R}_0)} \right)^{o} \tag{0.6}$$

is a fundamental domain, called *cuspidal standard fundamental domain*, for Γ.

It is not clear whether we can always choose the right coset representatives so that the cuspidal standard fundamental domain is h-convex. In this thesis I produced a counterexample.

In addition, it is not obvious whether we can choose the complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ so that the standard fundamental domain is h-convex. I have answered this question partially in this thesis. Moreover, the proof of the result mentioned above is algorithmic. I have also shown, using the method that I have developed, if we use $\mathcal{R}^0$ instead of $\mathcal{R}_0$ then we can always choose a complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ appropriately so that the set

$$\mathcal{R}^\Gamma = \left(\overline{\bigcup_{k=1}^{\mu} A_k(\mathcal{R}^0)} \right)^{o} \tag{0.7}$$

is an h-convex fundamental domain for Γ.

All the results mentioned above hold if we can replace $\Gamma(1)$ by the discrete Hecke groups $H(\lambda)$, where $\lambda = 2cos(\frac{\pi}{q})$ for some $q \in \mathbb{Z}$ and $q \geq 3$.

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DEDICATION

To my parents Yayenie Teka and Zeyneb Adem with all my love and gratitude.

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NOTATIONS

We write

- $\mathbb{C}$ for the set of complex numbers, with the usual topology.
- $\mathbb{R}$ for the set of real numbers.
- $\mathbb{Q}$ for the set of rational numbers.
- $\mathbb{Z}$ for the set of integers.
- $\mathbb{Z}^+$ for the set of positive integers.
- $\mathbb{H} := \{\tau \in \mathbb{C} : Im(\tau) > 0\}$ for the upper half-plane.
- $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, the extended complex plane. This is the one point compactification of $\mathbb{C}$. In the topology of $\mathbb{C}_\infty$, a set A is open if either (i) A is open subset of $\mathbb{C}$, or (ii) $\infty \in A$ and $\mathbb{C}_\infty - A$ is compact in $\mathbb{C}$. With this topology, $\mathbb{C}_\infty$ is homeomorphic to the Riemann sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.
- $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, the one point compactification of $\mathbb{R}$.
- $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R}_\infty$.
- $\mathbb{P} := \mathbb{Q} \cup \{\infty\}$.
- $\mathbb{H}' := \mathbb{H} \cup \mathbb{P}$.
- $\Theta := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} = SL(2, \mathbb{R})$.
- $\widetilde{\Gamma}(1) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} = SL(2, \mathbb{Z})$.
- $\mathcal{H} := \left\{ 2\cos(\frac{\pi}{q}) : q \in \mathbb{Z},\ q \geq 3 \right\}$.
- $\overline{A}$ and A^o denote the closure and interior of the set A, respectively.

CHAPTER 1

INTRODUCTION

By $H(\lambda)$ we shall mean the group of linear fractional transformations generated by the transformations,

$$S_\lambda : \tau \longrightarrow \tau + \lambda, \text{ and } T : \tau \longrightarrow \frac{-1}{\tau},$$

where λ is a fixed positive real number. Discrete $H(\lambda)$ are useful in the study of Dirichlet series with functional equations. It has been shown by Hecke in [8], and later by Evans in [5], that $H(\lambda)$ is discrete if and only if either $\lambda \geq 2$ or $\lambda = 2cos\left(\frac{\pi}{q}\right)$ for $q = 3, 4, 5, ...$ These discrete groups are called Hecke groups. It is easy to see that $\Gamma(1) = H(\lambda)$, for $\lambda = 2cos\left(\frac{\pi}{3}\right)$, where $\Gamma(1) = SL(2, \mathbb{Z})$. The theory of automorphic forms has been prominent in number theory since its inception. This theory builds upon classical methods and leads to deep insight into ancient problems. The theory of automorphic forms is the study of beautifully symmetric functions on hyperbolic space. The simplest symmetric functions are periodic functions(see [15]). A periodic function f of period β is a function which satisfies the functional equations

$$f(x + m\beta) = f(U^m(x)) = f(x), \forall x \ \in \mathbb{R} \ \& \ \forall m \in \mathbb{Z},$$

where U is translation by β. Therefore a periodic function of period β is a function which is invariant under the group $\langle U \rangle$. The concept of an automorphic function is a natural generalization of a periodic function. A function

$f : \mathbb{H} \to \mathbb{C}$ with the property

$$f(M(\tau)) = f(\tau), \quad \forall M \in \Gamma \ \& \ \forall \tau \in \mathbb{H},$$

is called **an automorphic function** on the group Γ. An **automorphic form** on the group Γ is a function $f : \mathbb{H} \to \mathbb{C}$ with the property

$$f(M(\tau)) = j_M(\tau) f(\tau), \quad \forall M \in \Gamma \ \& \ \forall \tau \in \mathbb{H},$$

where $j_M(\tau)$, a function of M and τ, satisfies a certain condition called the "consistency condition".

If one wants to investigate the properties of a periodic function of period β, it is sufficient to study the function on the fundamental period interval or strip $[0, \beta]$ and this makes life much easier. By the same token, to study automorphic forms on the group Γ, it is sufficient to study the function on the fundamental domain for Γ, which is analogous to the fundamental period strip for periodic functions.

Two major ways of obtaining fundamental domains for discrete subgroups of $SL(2, \mathbb{R})$ are the Dirichlet Polygon construction(see [15]) and Ford's construction(see [6]). Both methods give an h-convex fundamental domain for any discrete subgroup of $SL(2, \mathbb{R})$ as shown in Chapter 3.

But the Dirichlet polygon construction and Ford's construction are not well adapted for the actual construction of an h-convex fundamental domain. The reason for this will be clear in Chapter 3.

A third-and most important and practical-method of obtaining a fundamental domain is through the use of a right coset decomposition as described below.

The sets

$$\mathcal{R}_0 = \left\{ \tau \in \mathbb{H} : |\tau| > 1 \ \& \ |Re(\tau)| < \frac{1}{2} \right\} \tag{1.1}$$

and

$$\mathcal{R}^0 = \left\{ \tau \in \mathbb{H} : |\tau - 1| > 1 \ \& \ 0 < Re(\tau) < \frac{1}{2} \right\}, \tag{1.2}$$

are fundamental domains for $\Gamma(1)$. These sets are hyperbolically convex and simply connected domains in the upper half-plane. It is well known that if Γ is s subgroup of $\Gamma(1)$, and

$$\Gamma(1) = \Gamma \cdot \{A_1, A_2, \cdots, A_\mu\}, \tag{1.3}$$

then the set

$$\mathcal{R}_\Gamma = \left(\overline{\bigcup_{k=1}^{\mu} A_k(\mathcal{R}_0)}\right)^o \tag{1.4}$$

is a fundamental domain for Γ. The same is true if we replace $\mathcal{R}_0$ by $\mathcal{R}^0$. The above type of fundamental domain for Γ is called *a standard fundamental domain* for Γ. It is known that if $[\Gamma(1) : \Gamma] = \mu < \infty$, then there exists a finite number of elements $C_1, \cdots, C_p \in \Gamma(1)$ and $\lambda_1, \cdots, \lambda_p \in \mathbb{N}$ such that

$$\Gamma(1) = \Gamma \cdot \bigcup_{k=1}^{p} \{C_k, C_kS, ..., C_kS^{\lambda_k - 1}\}. \tag{1.5}$$

Applying the above result we find that the set

$$\mathcal{R}_\Gamma = \left(\overline{\bigcup_{k=1}^{p} \bigcup_{j=1}^{\lambda_k - 1} C_kS^j(\mathcal{R}_0)}\right)^o \tag{1.6}$$

is a fundamental domain for Γ. The fundamental domain for Γ given in (1.6) is called *cuspidal standard fundamental domain* for Γ and the elements $C_1, \cdots, C_p$ are called *cuspidal right coset representatives.* The integer p, given in (1.5), is the number of points of intersection of the closure of cuspidal standard fundamental domain and $\mathbb{R}_\infty$ and it is called the parabolic class number of the group.

It is not clear whether we can always choose the right coset representatives so that the cuspidal standard fundamental domain is hyperbolically convex. I show here that there is a counterexample. The group $\Gamma^0(q^3)$, q prime, is a subgroup of $\Gamma(1)$ such that $[\Gamma(1) : \Gamma^0(q^3)] = q^3 + q^2$, and $p = 2q$. I have proved that if $q \geq 5$, then we cannot choose the right coset representatives $C_1, \cdots, C_{2q}$

so that the cuspidal standard fundamental domain is hyperbolically convex, i.e, the subgroup $\Gamma^0(q^3)$ has no cuspidal standard fundamental domain which is hyperbolically convex.

In addition, it is not obvious whether we can choose the complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ so that the standard fundamental domain, given by the above theorem, is hyperbolically convex. I have answered this question partially in this thesis. The result can be stated as follows: If the subgroup Γ of $\Gamma(1)$ is a normal subgroup, then we can always choose a complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ so that the standard fundamental domain becomes hyperbolically convex. Moreover, from the proof of the above result(Theorem) I have developed an algorithm for constructing a set of right cosets Σ and hence a hyperbolically convex standard fundamental domain for Γ. I have implemented the above algorithmic construction for principal congruence subgroups and for normal subgroups of the modular group of genus 1. These latter groups are completely described by Newman [16].

I have also shown, using the method that I have developed in proving the above result that if we use $\mathcal{R}^0$ instead of $\mathcal{R}_0$(which avoids much of the difficulty we faced in using $\mathcal{R}_0$) in equation (1.4), then we can always choose a complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ appropriately so that the set

$$\mathcal{R}^\Gamma = \left(\overline{\bigcup_{k=1}^{\mu} A_k(\mathcal{R}^0)}\right)^o \tag{1.7}$$

is a hyperbolically convex fundamental domain for Γ. This latter result was originally proved by Kulkarni [14]. The main difficulty in using $\mathcal{R}_0$ is that this set has two elliptic points of order 3, while the set $\mathcal{R}^0$ contains only one elliptic point of order 3. Hence the set $\mathcal{R}^\Gamma$ has fewer elliptic vertices of order 3 than does the set $\mathcal{R}_\Gamma$ and therefore the former set has a better chance of being h-convex. The importance of using $\mathcal{R}_0$ instead of $\mathcal{R}^0$ rests upon-among other things-two facts:

(i) the standard fundamental domains constructed using the former are very useful in applications to number theory;

(ii) the standard fundamental domain constructed using the former will easily give us a set of generators for the group.

I have also considered the discrete Hecke groups $H(\lambda)$, where $\lambda = 2cos(\frac{\pi}{q})$ for some $q \in \mathbb{Z}$ and $q \geq 3$. The sets defined by

$$\mathcal{R}_\lambda = \left\{\tau \in \mathbb{H} : |\tau| > 1 \ \& \ |Re(\tau)| < \frac{\lambda}{2}\right\} \tag{1.8}$$

and

$$\mathcal{R}^\lambda = \left\{\tau \in \mathbb{H} : |\tau - \frac{1}{\lambda}| > \frac{1}{\lambda} \ \& \ 0 < Re(\tau) < \frac{\lambda}{2}\right\}, \tag{1.9}$$

are fundamental domains for $H(\lambda)$. These sets are hyperbolically convex and simply connected domains in the upper half-plane. It is also known that if Γ is a subgroup of $H(\lambda)$ and $H(\lambda) = \Gamma \cdot \{A_1, A_2, A_3, ...\} = \Gamma \cdot \Sigma$, then the set

$$\mathcal{R}_{\Gamma,\Sigma} = \left(\overline{\bigcup_{A\in\Sigma} A\left(\mathcal{R}_\lambda\right)}\right)^o \tag{1.10}$$

is a fundamental domain for Γ. The same is true if we replace $\mathcal{R}_\lambda$ by $\mathcal{R}^\lambda$. But it is not known whether the set $\mathcal{R}_{\Gamma,\Sigma}$ has a nice topological structure such as convexity in hyperbolic geometry and simple-connectedness. If Γ is a normal subgroup of $H(\lambda)$ of finite index, then I have proved here the existence of a complete right coset system Σ such that the set $\mathcal{R}_{\Gamma,\Sigma}$ is hyperbolically convex. I have also proved that if we use $\mathcal{R}^\lambda$ instead of $\mathcal{R}_\lambda$ in equation (1.10) and $[H(\lambda : \Gamma] = \mu$, then we can always choose a complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ appropriately so that the set

$$\mathcal{R}^{\Gamma,\Sigma} = \left(\overline{\bigcup_{k=1}^{\mu} A_k(\mathcal{R}^\lambda)}\right)^o \tag{1.11}$$

is a hyperbolically convex fundamental domain for Γ(Kulkarni does not address this issue in his long and well-known article [14]). As a byproduct of the above result I have given a generalization of Farey fractions, what I call the λ-Farey fractions.

CHAPTER 2

THE UPPER HALF-PLANE AND THE MÖBIUS TRANSFORMATIONS

2.1 Linear Fractional Transformations

Definition 2.1 *A linear fractional transformation, or Möbius transformation, is a nonconstant rational function*

$$L : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty, z \to w = L(z) := \frac{az+b}{cz+d}, \tag{2.1}$$

where a, b, c, d *are complex numbers satisfying* $ad - bc \neq 0$.

The usual convention about the linear fractional transformation defined in (2.1) must be adopted. That is,

$$L(\infty) = \frac{a}{c}, \text{and } L\left(\frac{-d}{c}\right) = \infty.$$

The references for the material in this chapter are [13], [15], [25], and [27]. The set of all linear fractional transformations shall be denoted by $M\ddot{o}b(\mathbb{C})$. One can easily show that the set $M\ddot{o}b(\mathbb{C})$ can be redefined as follows:

$$M\ddot{o}b(\mathbb{C}) = \left\{ L : \tilde{\mathbb{C}} \longrightarrow \tilde{\mathbb{C}} : \ L(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{C}, \quad ad - bc = 1 \right\}. \tag{2.2}$$

In this section we will see some of the fascinating properties of the elements of $M\ddot{o}b(\mathbb{C})$.

If $L(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation, then $L^{-1}(z) = \dfrac{dz-b}{-cz+a}$ satisfies $L(L^{-1}(z)) = L^{-1}(L(z)) = z$, that is L^{-1} is the inverse mapping of L and it belongs to $M\ddot{o}b(\mathbb{C})$. If L and M are two linear fractional transformations, then their composition $L \circ M$ is also a linear fractional transformation. Therefore the set of Möbius transformations forms a group under composition.

Theorem 2.1 *If L is a Möbius transformation, then L is the composition of a translation $z+\alpha$, a dilation βz, $\beta > 0$, a rotation $e^{i\theta}z$, and an inversion $\frac{1}{z}$.*

Proof: We consider a transformation $L \in M\ddot{o}b(\mathbb{C})$ such that

$$L(z) \;=\; \frac{az+b}{cz+d}.$$

If $c = 0$, then $L(z) = \frac{a}{d}z + \frac{b}{d}$ which is a composition of a dilation, a rotation and a translation.

Let $c \neq 0$ and put $L_1(z) = z + \frac{d}{c}$, $L_2(z) = \frac{1}{z}$, $L_3(z) = z + \frac{b}{d}$, and $L_4(z) = z + \frac{a}{c}$. Then we can easily see that $L = L_4 \circ L_3 \circ L_2 \circ L_1$. ■

Theorem 2.2 *The linear fractional transformation (2.1) is a one-to-one conformal mapping of $\mathbb{C}_\infty$ on itself.*

The proof of this theorem is evident from Theorem 2.1 and hence we will not include it.

Let $\psi : SL(2,\mathbb{C}) \to M\ddot{o}b(\mathbb{C})$ be defined by

$$\psi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \;=\; L \;\text{ where } L(z) \;=\; \frac{az+b}{cz+d} \tag{2.3}$$

then we can easily see that ψ is an onto group homomorphism and

$$Ker(\psi) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \{I, -I\} \tag{2.4}$$

Therefore $SL(2,\mathbb{C})/\{I,-I\} \cong M\ddot{o}b(\mathbb{C})$ and hence we can identify $M\ddot{o}b(\mathbb{C})$ with $SL(2,\mathbb{C})$ provided we identify $M \in SL(2,\mathbb{C})$ with $-M$.

One of the main reasons people study Möbius transformations in hyperbolic geometry is contained in the following theorem.

Theorem 2.3 *A Möbius transformation maps circles in $\mathbb{C}_\infty$ onto circles in $\mathbb{C}_\infty$.*

Proof: By Theorem 2.1 it is enough to prove that the inversion, $L(z) = \frac{1}{z}$ maps circles into circles. Let Ω be a circle. Since the equation

$$Az\bar{z} + Bz + \bar{B}z + C = 0, \tag{2.5}$$

where $A, C \in \mathbb{R}$, includes all circles, Ω is given by

$$Az\bar{z} + Bz + \bar{B}z + C = 0, \tag{2.6}$$

for some $A, C \in \mathbb{R}$, and $B \in \mathbb{C}$. Then if $w = L(z) = \frac{1}{z}$, $L(\Omega)$ is given by the equation

$$Cw\bar{w} + B\bar{w} + \bar{B}w + A = 0. \tag{2.7}$$

This shows that $L(\Omega)$ is also a circle.■

2.2 Fixed Points

Definition 2.2 *A point z is called a fixed point of a map f if $f(z) = z$.*

Here we are interested only in the fixed points of nonidentity Möbius transformations. Let $L(z) = \frac{az+b}{cz+d}$ be a nonidentity Möbius map. If $L(z) = z$, then

$$cz^2 + (d-a)z - b = 0$$

Since a polynomial of degree two can't have more than two roots, a Möbius map L can have at most two fixed points. If $c = 0$ and $a = d$, then the set of fixed point is $\{i\infty\}$, and if $c = 0$ and $a \neq d$, then the set of fixed points is $\{i\infty, \frac{b}{d-a}\}$.

If $c \neq 0$, then

$$L(z) = z \Longleftrightarrow cz^2 + (d-a)z - b = 0$$

$$\Longleftrightarrow z_1, z_2 = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

It is clear that the nature of the roots z_1, z_2 depends upon the number $(a+d)^2 - 4 = trace(L)^2 - 4$, where $trace(L) := a + d$.

The following lemma follows from the fact that the trace of a matrix is invariant under conjugation by any invertible matrix over the same field.

Lemma 2.1 *The trace of a Möbius transformation is invariant under conjugation by elements of $M\ddot{o}b(\mathbb{C})$.*

Remark 2.1 *The two fixed points of a Möbius map L are coincident(equal) if and only if $trace(L) = \pm 2$, and in this case the transformation is called parabolic.*

If $L \in M\ddot{o}b(\mathbb{C})$ is not the identity map, then L has at most two fixed points. It follows that $L \in M\ddot{o}b(\mathbb{C})$ is uniquely determined by the images of three distinct points of $\mathbb{C}_\infty$. These may be specified arbitrarily:

$$z_1 \to w_1, \quad z_2 \to w_2, \quad z_3 \to w_3,$$

and the map may be realized by

$$\left(\frac{L(z) - w_1}{L(z) - w_2}\right)\left(\frac{w_3 - w_2}{w_3 - w_1}\right) = \left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right), \tag{2.8}$$

where suitable modifications must be made for the appearance of ∞. The expression

$$\left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right) =: (z; z_1, z_2, z_3) \tag{2.9}$$

is called the *cross ratio* of the four points $z, z_1, z_2,$ and z_3. One other important fact about the Möbius transformations is that they preserve the cross ratio of four points,i.e., if $N \in M\ddot{o}b(\mathbb{C})$, then $(N(z); N(z_1), N(z_2), N(z_3)) = (z; z_1, z_2, z_3)$.

The following theorem, which follows directly from the above discussion, is a basis for the classification of Möbius transformation into parabolic, elliptic, hyperbolic and loxodromic types.

Theorem 2.4 *Let $L \in M\ddot{o}b(\mathbb{C})$. Then*

1. *If L has two distinct fixed points $\zeta_1, \zeta_2 \in \mathbb{C}$, then*

$$\frac{L(z)-\zeta_1}{L(z)-\zeta_2} = \kappa\frac{z-\zeta_1}{z-\zeta_2}, \quad \kappa \in \mathbb{C}, \quad \kappa \neq 0,\ 1$$

2. *If L has two distinct fixed points $\zeta_1 \in \mathbb{C}$, and $i\infty$, then*

$$L(z) - \zeta_1 = \kappa(z-\zeta_1), \quad \kappa \in \mathbb{C}, \quad \kappa \neq 0,\ 1$$

3. *If L has only one fixed point $\zeta_1 \in \mathbb{C}$, then*

$$\frac{1}{L(z)-\zeta_1} = \frac{1}{z-\zeta_1} + \alpha, \quad \alpha \in \mathbb{C}, \quad \alpha \neq 0, \quad \text{and} \quad \kappa = 1.$$

4. *If L has $i\infty$ as the only fixed point, then*

$$L(z) = z + \alpha, \quad \alpha \in \mathbb{C}, \ \alpha \neq 0, \quad \kappa = 1$$

Definition 2.3 *For any $L \in M\ddot{o}b(\mathbb{C})$, the description of the above theorem is called the normal form of L and the number $\kappa + \frac{1}{\kappa}$ is called the multiplier of L.*

Lemma 2.2 *Let $L \in M\ddot{o}b(\mathbb{C})$ and $\kappa + \frac{1}{\kappa}$ be the multiplier of L. Then $\kappa + \frac{1}{\kappa}$ is also the multiplier of ALA^{-1} for any $A \in M\ddot{o}b(\mathbb{C})$. Moreover $\kappa + \frac{1}{\kappa} = trace(L)^2 - 2$.*

Proof: First let us assume that L has two distinct fixed points in $\mathbb{C}$ and

$$L(z) = \frac{az+b}{cz+d}.$$

Then the fixed points of L are $z_1 = \dfrac{a-d-\sqrt{(a+d)^2-4}}{2c}$, and $z_2 = \dfrac{a-d+\sqrt{(a+d)^2-4}}{2c}$. Clearly κ is given by $\kappa = \frac{a+d+\sqrt{(a+d)^2-4}}{a+d-\sqrt{(a+d)^2-4}}$ and hence

$$\kappa + \frac{1}{\kappa} = \frac{a+d+\sqrt{(a+d)^2-4}}{a+d-\sqrt{(a+d)^2-4}} + \frac{a+d-\sqrt{(a+d)^2-4}}{a+d+\sqrt{(a+d)^2-4}} = trace(L)^2 - 2.$$

Similarly one can show that the above formula holds for any Möbius transformation. Since trace is invariant under conjugation, $\kappa + \frac{1}{\kappa}$ is invariant under conjugation. ■

Since $\kappa \in \mathbb{C}$, we can write κ as

$$\kappa = \delta e^{i\theta}, \quad \delta > 0, \quad 0 \leq \theta < 2\pi \tag{2.10}$$

We can classify members of $M\ddot{o}b(\mathbb{C})$ as follows:

- $\delta = 1, \ \theta \neq 0$, L is called elliptic
- $\delta \neq 1, \ \theta = 0$, L is called hyperbolic
- $\delta \neq 1, \ \theta \neq 0$, L is called loxodromic
- $\delta = 1, \ \theta = 0$, L is called parabolic

We can summarize the above discussion in a single theorem given below.

Theorem 2.5 *A necessary and sufficiency condition that $L \in M\ddot{o}b(\mathbb{C})$ be elliptic, hyperbolic or parabolic is that trace(L) be real and $|trace(L)| < 2$, $|trace(L)| > 2$, or $|trace(L)| = 2$, respectively. Moreover, this classification is invariant under conjugation by an element of $M\ddot{o}b(\mathbb{C})$.*

In this thesis we are interested only in linear fractional transformations that map the upper half-plane $\mathbb{H}$ onto itself. If $L \in M\ddot{o}b(\mathbb{C})$ maps $\mathbb{H}$ into itself, then L must map $\mathbb{R}_\infty$ onto itself and $\mathrm{Im}(L(i)) > 0$. Let us define a subset of $M\ddot{o}b(\mathbb{C})$ by

$$M\ddot{o}b(\mathbb{R}) = \left\{ L \in M\ddot{o}(\mathbb{C}) : \ L(z) = \frac{az+b}{cz+d}, \ a,b,c,d \in \mathbb{R} \ ad-bc = 1 \right\}. \tag{2.11}$$

Theorem 2.6 *$L \in Möb(\mathbb{C})$ preserves $\mathbb{H}$ if and only if $L \in Möb(\mathbb{R})$.*

Proof: ($\Longleftarrow$) Suppose that $L \in Möb(\mathbb{R})$. we want to show that $L(\mathbb{H}) = \mathbb{H}$. Since $L(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, then

$$Im(L(\tau)) = \frac{Im(\tau)}{|c\tau + d|^2} > 0, \ \ \forall \tau \in \mathbb{H}.$$

Therefore $L(\mathbb{H}) \subset \mathbb{H}$. Since L is invertible and its inverse is also in $Möb(\mathbb{R})$, then $\mathbb{H} \subset L^{-1}(\mathbb{H}) \subset \mathbb{H}$. Therefore $L(\mathbb{H}) = \mathbb{H}$.

($\Longrightarrow$) Suppose that $L \in Möb(\mathbb{C})$ and $L(\mathbb{H}) = \mathbb{H}$. We want to show that $L \in Möb(\mathbb{R})$.

$$\begin{aligned} x \in \mathbb{R} &\Longrightarrow L(x) \in \mathbb{R} \\ &\Longrightarrow \overline{L(x)} = L(x) \\ &\Longrightarrow \overline{\left(\frac{ax+b}{cx+d}\right)} = \frac{ax+b}{cx+d} \\ &\Longrightarrow \frac{\bar{a}x+\bar{b}}{\bar{c}x+\bar{d}} = \frac{ax+b}{cx+d} \\ &\Longrightarrow \bar{L}(z) = L(z), \ \ \forall z \in \mathbb{H} \\ &\Longrightarrow \bar{L} = L \\ &\Longrightarrow \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

If $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = - \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we can easily see that

$$Im(L(i)) = \frac{-1}{|ci + d|^2} < 0$$

and this contradicts the hypothesis. Therefore $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus, $L \in Möb(\mathbb{R})$. ■

From now on we consider only linear transformations that preserve $\mathbb{H}$, that is, members of the group $Möb(\mathbb{R})$ and later we will specialize to subgroups of $M\ddot{o}b(\mathbb{R})$. As we did before, we can identify $M\ddot{o}b(\mathbb{R})$ with $SL(2,\mathbb{R})$ by identifying $M \in SL(2,\mathbb{R})$ with $-M$. The next section contains some of the properties of $SL(2,\mathbb{R})$ and its subgroups. Before we go to the next section let us give one characterization of commutating elements of the group $M\ddot{o}b(\mathbb{R})$.

Theorem 2.7 *Two real linear fractional transformations, neither of which is the identity, commute if and only if they have the same set of fixed points.*

2.3 Discrete Subgroups of $SL(2,\mathbb{R})$

Definition 2.4 *A group $\Gamma \subset SL(2,\mathbb{R})$ is called discrete if it contains no convergent sequence of distinct matrices. By convergence we mean entrywise convergence.*

From the definition, any finite subgroup of $SL(2,\mathbb{R})$ is discrete and any subgroup of a discrete group is also discrete.

Lemma 2.3 *A group Γ is discrete if and only if there is no sequence $M_n \to I$ where $\{M_n\}$ consists of distinct elements.*

Proof: Suppose that there is no distinct sequence $\{M_n\}$ in Γ converging to I. We want to show that Γ is discrete. Assume that there is a distinct sequence $\{M_n\}$ in Γ converging to M. Then $M_n^{-1} \to M^{-1}$. Therefore

$$M_{n+1}M_n^{-1} \to I$$

If we show that the sequence $\{M_{n+1}M_n^{-1}\}$ contains infinitely many distinct elements then we are done. Hence it is enough to show that the above sequence contains infinitely many distinct elements. If $\{M_{n+1}M_n^{-1}\}$ contains only finitely many distinct elements, then there exists a subsequence $\{M_{n_k+1}M_{n_k}^{-1}\}$ which are identical. Since $\{M_{n+1}M_n^{-1}\} \to I$, then $\{M_{n_k+1}M_{n_k}^{-1}\} \to I$. That means $M_{n_k+1}M_{n_k}^{-1} = I \quad \forall k \in \mathbb{N}$. Hence $M_{n_k+1} = M_{n_k} \quad \forall k \in \mathbb{N}$. This is a contradiction. The converse of the statement is not difficult to see. ■.

We can characterize discrete groups in another way. To do that we need the following useful terminology.

Definition 2.5 *Let Γ be a subgroup of $SL(2,\mathbb{R})$. The stabilizer of a point τ with respect to the group Γ, denoted by Γ_τ, is defined by*

$$\Gamma_\tau := \{M \in \Gamma : M(\tau) = \tau\}$$

We can easily check that $\Gamma_{M(\tau)} = M\Gamma_\tau M^{-1}$ for any $M \in SL(2,\mathbb{R})$. Moreover a group Γ is discrete if and only if for all $\tau \in \mathbb{H}$ Γ_τ has no limit point if and only if the orbit $[\tau]$ has no accumulation point in $\mathbb{H}$.

The following theorem is very useful for our discussion.

Theorem 2.8 *Let Γ_2 be a subgroup of finite index μ in a group Γ_1 and let N be a fixed element of Γ_1. Then there exist a finite number of elements $M_1, M_2, \ldots, M_t$ in Γ_1 and t disjoint sets*

$$\mathcal{M}_k := \left\{M_k, M_kN, \ldots, M_kN^{\lambda_k-1}\right\}, \quad k = 1, 2, \ldots, t \tag{2.12}$$

where

$$\lambda_k = min\left\{l \in \mathbb{N} : N^l \in M_k^{-1}\Gamma_2 M_k\right\} \tag{2.13}$$

such that

(i) $\mu = \lambda_1 + \lambda_2 + \cdots + \lambda_t$

(ii) $\Gamma_1 = \Gamma_2 \cdot \bigcup_{k=1}^{t} \mathcal{M}_k$.

Moreover, if N has finite order σ, then λ_k divides σ for all $1 \leq k \leq t$. Also, if Γ_2 is normal subgroup of Γ_1, then $\lambda_k = \lambda_1$ $\forall k$ and $\mu = t\lambda_1$.

Proof: Take any $M_1 \in \Gamma_1$ and define λ_1 by (2.13); since $M_1^{-1}\Gamma_2 M_1$ has finite index μ in Γ_1, λ_1 is a finite positive number and the members of $\mathcal{M}_1$ belong to λ_1 different right cosets of Γ_2 in Γ_1. If $\mu = \lambda_1$, this completes the proof and $m = 1$ in this case. If $\mu > \lambda_1$, we take M_2 not belonging to $\Gamma_2\mathcal{M}_1$ and define λ_2 by (2.13). As before the λ_2 elements M_2N^k $(0 \leq k < \lambda_2)$ belong to different right cosets of Γ_2. Moreover, $M_2N^k \notin \Gamma_2\mathcal{M}_1$; for if $M_2N^k \in \Gamma_2\mathcal{M}_1$ then $M_2 \in \Gamma_2\mathcal{M}_1N^{-k} = \Gamma_2\mathcal{M}_1$, which is false. If $\mu = \lambda_1 + \lambda_2$ the theorem follows; if $\mu > \lambda_1 + \lambda_2$, we take an $M_3 \notin \Gamma_2(\mathcal{M}_1 \cup \mathcal{M}_2)$ and proceed similarly.

Since μ is finite and $\lambda_k > 0$ for each k, the process terminates after a finite number of steps giving the required result. The remainder of the theorem is an immediate consequence.

2.4 The Modular Group

Notation

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$P = TS = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

We can easily see that $P^3 = T^2 = -I$ and $S^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. As mappings T and P have order 2 and 3 respectively. Let

$$\Gamma_1 = \langle S, T \rangle . \qquad (2.14)$$

Theorem 2.9 $SL(2,\mathbb{Z}) = \langle S, T \rangle = \Gamma_1$.

Proof: Since $S, T \in SL(2,\mathbb{Z})$, then $\Gamma_1 \subset SL(2,\mathbb{Z})$. We shall show that $SL(2,\mathbb{Z}) \subset \Gamma_1$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. If $a = 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} = \pm TS^d \in \Gamma_1.$$

If $b = 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \pm TS^{-c}T \in \Gamma_1.$$

Therefore we may assume that $a \neq 0$ and $b \neq 0$. Note that for $t \in \mathbb{Z}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} =: \begin{pmatrix} a & b' \\ c & d' \end{pmatrix},$$

where $b' = at + b, \ \ d' = ct + d$. Choose $t \in \mathbb{Z}$ so that $|b'| < |a|$(this is always possible). If $b' = 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b' \\ c & d' \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \in \Gamma_1.$$

If $b' \neq 0$, let $q \in \mathbb{Z}$ and consider

$$T \begin{pmatrix} a & b' \\ c & d' \end{pmatrix} T \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} =: \begin{pmatrix} -d' & c' \\ b' & -a' \end{pmatrix},$$

where $c' = c - d'q, \ \ a' = a - b'q$. Choose $q \in \mathbb{Z}$ so that $|a'| < |b'|$. Thus we have

$$T \begin{pmatrix} -d' & c' \\ b' & -a' \end{pmatrix} T = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Z}),$$

with $|a'| < |b'| < |a|$. If $a' = 0$, then as before $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_1$. If $a' \neq 0$, we repeat the entire process to obtain

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in SL(2, \mathbb{Z})$$

such that

$$|a''| < |a'| < |a|, \quad \text{or } b'' = 0.$$

Since $a, a', a'' \in \mathbb{Z}$ we eventually get an element in $SL(2, \mathbb{Z})$ of one of the two forms $\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ or $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$. Both are in Γ_1. Since all multiplications are by elements of Γ_1, we conclude that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. Therefore $SL(2, \mathbb{Z}) \subset \Gamma_1$. Thus,

$$SL(2, \mathbb{Z}) = \langle S, T \rangle \tag{2.15}$$

■

Let $\Gamma(1) = \left\{L \in M\ddot{o}b(\mathbb{R}) : L(\tau) = \frac{a\tau+b}{c\tau+d},\ a, b, c, d \in \mathbb{Z}\right\}$. Then we can easily see that $SL(2,\mathbb{Z})/\{I,-I\} \cong \Gamma(1)$. The group $\Gamma(1)$ is called the inhomogeneous modular group or simply the modular group and the group $SL(2,\mathbb{Z})$ is called the homogeneous modular group. From the above theorem we get

$$\Gamma(1) = \langle S, T\rangle, \tag{2.16}$$

where $S : \tau \to \tau + 1$, and $T : \tau \to \frac{-1}{\tau}$.

Using the above theorem we conclude that if $M \in SL(2,\mathbb{Z})$, then

$$M = S^{q_0}TS^{q_1}T\cdots TS^{q_n}, \tag{2.17}$$

where $q_k \in \mathbb{Z}$ for all $0 \le k \le n$. Since

$$I = STSTSTSTSTST = TSTSTSTSTSTS,$$

the representation 2.17 is not unique. Because of this we need another representation.

Theorem 2.10 *$SL(2,\mathbb{Z})$ is generated by T and P^2 and every element $M \in SL(2,\mathbb{Z})$ can be written uniquely in the form*

$$M = (-1)^r P^{2q_0}TP^{2q_1}T\cdots TP^{2q_n}, \tag{2.18}$$

where $0 \le r \le 1,\ 0 \le q_k \le 2(1 \le k \le n),\ \ q_k > 0(0 < k < n)$.

Proof: The only thing we have to show is uniqueness. The above representation is unique if we have

$$I = (-1)^r P^{2q_0}TP^{2q_1}T\cdots TP^{2q_n}, \tag{2.19}$$

only when $n = 0 = r = q_0$. Suppose that (2.19) holds for some $n > 0$ and assume that n is the least positive integer for which (2.19) holds. Then

$$I = (-1)^r TP^{2q_1}T\cdots TP^{2q_n+2q_0}.$$

If $q_n + q_0 \equiv 0 (\text{mod } 3)$, then $P^{2q_n+2q_0} = I$ and hence

$$I = (-1)^{r-1} P^{2q_1} T \cdots T P^{2q_{n-1}},$$

which is a word of length $n-2$. By assumption $n-2=0$ and so $n=2$. Hence

$$I \;=\; (-1)^r T P^{2q_1} T.$$

Therefore $I \;=\; (-1)^r P^{2q_1}$, and as a result, $q_1 \;=0,$ and $r = 0$. This is a contradiction. Therefore $q_0 + q_n \not\equiv 0 (\text{ mod } 3)$. In this case

$$\pm I = T P^{2q_1} T \cdots T P^{2q_n + 2q_0}. \tag{2.20}$$

That means $\pm I$ is a product of n factors each of which is either

$$TP^4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad -TP^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Since the entries in TP^4 and $-TP^2$ are all nonnegative the same is true of the product, which must be I. But if

$$I = TP^4 \cdot M, \quad \text{or} \quad I = (-TP^2) \cdot M,$$

then M must have a negative entry. This is a contradiction. Therefore the above representation is unique.

Definition 2.6 *For each $M \in SL(2,\mathbb{Z})$, expressed as in (2.18), we define*

$$\phi(M) \;=\; n + 2r,$$

and

$$\varphi(M) := q_0 + q_1 + \cdots + q_n.$$

Theorem 2.11 *The maps ϕ, $\varphi : SL(2,\mathbb{Z}) \longrightarrow \mathbb{Z}$ are group homomorphisms of $SL(2,\mathbb{Z})$ onto the additive groups of residue classes modulo 4 and 3 respectively.*

Proof: Because of the uniqueness of the representation (2.18) the maps ϕ and φ are well defined maps. We show that ϕ is a group homomorphism as described above and the other will follow similarly . The Let M and N be two elements of $SL(2,\mathbb{Z})$ expressed as

$$M = (-1)^r P^{2q_0} T P^{2q_1} T \cdots T P^{2q_m}, \tag{2.21}$$

$$N = (-1)^s P^{2t_0} T P^{2t_1} T \cdots T P^{2t_n}. \tag{2.22}$$

Then $\phi(M) = m + 2r$ and $\phi(N) = n + 2s$. Note that if $u \equiv r + s(mod 2)$, $0 \leq u \leq 1$, then

$$MN = (-1)^u P^{2q_0} T P^{2q_1} T \cdots T \underbrace{P^{2q_m} P^{2t_0}} T P^{2t_1} T \cdots T P^{2t_n}. \tag{2.23}$$

If $q_m + t_n \equiv v(\bmod 3)$, where $0 < v \leq 2$, then the unique representation of MN is given by

$$MN = (-1)^u P^{2q_0} T P^{2q_1} T \cdots T P^{2v} T P^{2t_1} T \cdots T P^{2t_n}, \tag{2.24}$$

and hence $\phi(MN) = n + m + 2u \equiv n + m + 2(r + s)(\bmod 4) \equiv \phi(M) + \phi(N)(\bmod 4)$.

Now suppose that $q_m + t_n \equiv 0(\bmod 3)$. Then

$$MN = (-1)^u P^{2q_0} T P^{2q_1} T \cdots T P^{2q_{m-1}} \underbrace{T P^{2q_m} P^{2t_0} T}_{-I} P^{2t_1} T \cdots T P^{2t_n}, \tag{2.25}$$

so that

$$MN = (-1)^{u+1} P^{2q_0} T P^{2q_1} T \cdots T P^{2q_{m-1}+2t_1} T \cdots T P^{2t_n}. \tag{2.26}$$

If $q_{m-1} + t_1 \not\equiv 0(\bmod 3)$, then $\phi(MN) \equiv n + m + 2u \pmod 4 \equiv n + m + 2(r + s) \pmod 4 \equiv \phi(M) + \phi(N)(\bmod 4)$. Otherwise,

$$MN = (-1)^{u+2} P^{2q_0} T P^{2q_1} T \cdots T P^{2q_{m-2}+2t_2} T \cdots T P^{2t_n}, \tag{2.27}$$

and we continue the process until it stops. At each stage of the process we get $\phi(MN) \equiv n+m+2u \pmod 4$ and therefore $\phi(MN) \equiv \phi(M)+\phi(N) \pmod 4$. Similarly we can show that $\varphi(MN) \equiv \varphi(M) + \varphi(N) \pmod 3$.■

Let

$$\Gamma^4 = Ker(\phi) = \{M \in SL(2,\mathbb{Z}) : \phi(M) \equiv 0(\bmod 4)\}$$

and

$$\Gamma^3 = Ker(\varphi) = \{M \in SL(2,\mathbb{Z}) : \varphi(M) \equiv 0(\bmod 3)\}.$$

By one of the isomorphism theorems Γ^3 and Γ^4 are normal subgroups of $SL(2,\mathbb{Z})$. Moreover,

$$SL(2,\mathbb{Z})/\Gamma^4 \cong \mathbb{Z}_4;$$

and

$$SL(2,\mathbb{Z})/\Gamma^3 \cong \mathbb{Z}_3.$$

Now we have all the tools needed to investigate the commutator subgroup of $SL(2,\mathbb{Z})$, denoted by $SL(2,\mathbb{Z})'$. This group is the smallest group containing all the commutators $[A,B]$ where $A, B, \in SL(2,\mathbb{Z})$ and

$$[A,B] = ABA^{-1}B^{-1}.$$

Theorem 2.12 *$SL(2,\mathbb{Z})'$ is generated by two elements*

$$A := STS^{-1}T, \text{ and } B := TS^{-1}TS \tag{2.28}$$

Furthermore $SL(2,\mathbb{Z})' = \Gamma^3 \cap \Gamma^4$, and $SL(2,\mathbb{Z})/SL(2,\mathbb{Z})'$ is a cyclic group generated by $SL(2,\mathbb{Z})' \cdot S$ of order 12.

Proof: Let $L, M \in SL(2,\mathbb{Z})$. Since $I = M \cdot M^{-1} = L \cdot L^{-1}$, by Theorem 2.11 we have

$$\phi(M) + \phi(M^{-1}) \equiv 0 \ (mod 4) \equiv \phi(L) + \phi(L^{-1}).$$

Hence

$$\phi([L,M]) \equiv \phi(L) + \phi(M) + \phi(L^{-1}) + \phi(M^{-1}) \equiv 0 \ (mod 4)$$

Similarly, $\varphi([L,M]) \equiv 0 \ (mod 3)$. Therefore if $N \in SL(2,\mathbb{Z})'$, then $N \in \Gamma^3 \cap \Gamma^4$.

Conversely, if $M \in \Gamma^3 \cap \Gamma^4$, then we can express M as follows:

$$M = (-1)^r P^{2q_0} T P^{2q_1} T \cdots T P^{2q_m}, \tag{2.29}$$

where $0 \leq r \leq 1$, $0 \leq q_k \leq 2(1 \leq k \leq n)$, $q_k > 0(0 < k < n)$. Let $t_k := q_0 + \ldots + q_k$ $(0 \leq k \leq m)$. Note that m must be even and we can write M as

$$M = (-1)^r [P^{2t_0}, T][T, P^{2t_1}] \cdots [P^{2t_{m-2}}, T][T, P^{2t_{m-1}}]. \qquad (2.30)$$

This shows that $M \in SL(2,\mathbb{Z})'$. Therefore $SL(2,\mathbb{Z})' = \Gamma^3 \cap \Gamma^4$. Since

$$[T, P^2] = [P^2, T]^{-1} = TS^{-1}TS = B,$$

and

$$[T, P^4] = [P^4, T] = STS^{-1}T = A,$$

then $SL(2,\mathbb{Z})'$ is generated by A and B. Here we need one result from group theory. If H, and K are two subgroups of G such that $([G : H], [G : K]) = 1$, then $[G : H \cap K] = [G : H][G : K]$(see any abstract algebra book). Therefore

$$[SL(2,\mathbb{Z}) : SL(2,\mathbb{Z})'] = [SL(2,\mathbb{Z}) : \Gamma^3 \cap \Gamma^4] = [SL(2,\mathbb{Z}) : \Gamma^3][SL(2,\mathbb{Z}) : \Gamma^4] = 12.$$

Now it remains to show that $SL(2,\mathbb{Z})' \cdot S$ has order 12. For any $k \in \mathbb{N}$, we have

$$S^k = \underbrace{P^0 T P^4 T P^4 T \cdots T P^4}_{\text{word of length k}}.$$

Hence $\varphi(S^k) = 2k$ and $\phi(S^k) = k$. The smallest natural number k for which the matrix S^k is in the commutator subgroup $SL(2,\mathbb{Z})'$ is $k = 12$.■

We now state the corresponding results for the associated inhomogeneous modular group. Note that $-I \notin \Gamma^4$. Let $\Gamma(1)^4 \cong \Gamma^4$, $\Gamma(1)^3 \cong \Gamma^3/\{I, -I\}$, and $\Gamma(1)'$ be the commutator subgroup of the group $\Gamma(1)$.

Theorem 2.13 $\Gamma(1)' = \Gamma(1)^3 \cap \Gamma(1)^4$, $[\Gamma(1) : \Gamma(1)'] = 6$ *and* $\Gamma(1)/\Gamma(1)'$ *is a cyclic group of order 6 generated by the coset containing the translation* S.

2.4.1 Normal subgroups of $\Gamma(1)$ contained in the Commutator subgroup

Here I will mention the most exciting result by Morris Newman [16] about normal subgroups of $\Gamma(1)$ which are contained in the commutator subgroup

$\Gamma(1)'$. As we have seen before, any element of $\Gamma(1)'$ can be written as a word in $A = STS^{-1}T$ and $B = TS^{-1}TS$. Let $e_A(M)$ and $e_B(M)$ denote the sum of the exponents of A and B, respectively.

Theorem 2.14 *For each triplet of integers (p, m, d) such that p positive, $0 \leq m \leq d-1$, and $m^2 + m + 1 \equiv 0(\ mod\ d)$, the set*

$$\Gamma(p, m, d) = \left\{ M \in \Gamma(1)' : e_A(M) \equiv 0 \ (mod\ p), \ \ e_B(M) \equiv m e_A(M) \ (mod\ dp) \right\} \tag{2.31}$$

is a normal subgroup of the modular group $\Gamma(1)$ of index $6dp^2$.

2.4.2 Congruence Subgroups of the modular group

Definition 2.7 *Let n be a natural number. We define the principal congruence subgroup of level n, by*

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod n}) \right\}. \tag{2.32}$$

Let $\Gamma_0(n)$ be the subgroup of $\Gamma(1)$ defined by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : \ n | c \right\} \tag{2.33}$$

and let $\Gamma^0(n)$ be the subgroup of $\Gamma(1)$ defined by

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : \ n | b \right\}. \tag{2.34}$$

Any subgroup of the modular group which contains $\Gamma(n)$, for some $n \in \mathbb{N}$, is called a congruence subgroup of level n.

The congruence subgroup $\Gamma_0(n)$ is called the Hecke congruence group and is often encountered in arithmetic.

Theorem 2.15 *Let n be a natural number. Then* *(i)* $[\Gamma(1):\Gamma(n)] = \begin{cases} 6 & ;\text{ifn} = 2 \\ \frac{n^3}{2}\prod_{p|n}\left(1-\frac{1}{p^2}\right) & ;\text{ifn} \geq 3 \end{cases}$ *and* $[\Gamma(1):\Gamma_0(n)] = n\prod_{p|n}\left(1+\frac{1}{p}\right)$.

(ii) A set of representatives for $\Gamma_0(n)\backslash\Gamma(1)$ *is given by*

$$\begin{pmatrix} * & * \\ u & v \end{pmatrix} \in \Gamma(1) \quad \textit{with } v|n, \quad u\,(\textit{mod } \frac{n}{v}). \tag{2.35}$$

(iii) A set of inequivalent cusps for $\Gamma_0(n)$ *is given by the following fractions*

$$\frac{u}{v} \quad \textit{with } v|n, \quad (u,v)=1, \quad u\,(\textit{mod } (n/v,v)) \tag{2.36}$$

and the number of inequivalent cusps, denoted by $\sigma_\infty(n)$, *is* $\sigma_\infty(n) = \prod_{k=1}^{r}\left(p_k^{\lfloor\frac{t_k}{2}\rfloor} + p_k^{\lfloor\frac{t_k-1}{2}\rfloor}\right)$, *where* $n = \prod_{k=1}^{r} p_k^{t_k}$.

Proof: The proofs of (i) and (ii) are standard, but long, and can be found in the book of Gunning [7] and Iwaniec [10]. We shall provide the proof of (iii) using (ii). Suppose that $\frac{u}{v}$, $(u,v)=1$, $v|n$ and $\frac{u'}{v'}$, $(u',v')=1$, $v'|n$ are two cusps that are equivalent under $\Gamma_0(n)$. Then there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(n)$, such that

$$\frac{u'}{v'} = \frac{\alpha u + \beta v}{\gamma u + \delta v}.$$

Then $u' = \alpha u + \beta v$, $v' = \gamma u + \delta v$, $n|\gamma$ and hence $v|v'$. Similarly we can show that $v'|v$. Therefore $v = v'$. Since $v' = \gamma u + \delta v = \gamma u + \delta v'$, we get $(\delta - 1)v = \gamma u$ and this implies that $\delta - 1 \equiv 0(\textit{mod } \frac{n}{v})$. Hence

$$u' = \alpha u + \beta v \equiv \alpha u \equiv \alpha\delta u \equiv u(\textit{mod}(v, n/v)).$$

Therefore a set of inequivalent cusps for $\Gamma_0(n)$ is given by the following fractions

$$\frac{u}{v} \text{ with } v|n, \quad (u,v)=1, \quad u\,(\textit{mod } (n/v,v)). \tag{2.37}$$

Hence, $\sigma_\infty(n) = \sum_{v|n} |\{u\,(\textit{mod }(v,n/v)) : (u,v)=1\}|$. Let $n = p^t n_0$, where $(n_0,p)=1$. Then,

$$\begin{aligned}
\sigma_\infty(n) &= \sum_{v|n} |\{u \ (mod \ (v, n/v)) : (u, v) = 1\}| \\
&= \sum_{e=0}^{t} \sum_{s|n_0} |\{u \ (mod \ (p^e s, p^t n_0/p^e s)) : (u, p^e s) = 1\}| \\
&= \sum_{e=0}^{t} \sum_{s|n_0} \left|\{u \ (mod \ p^{min(e,t-e)}(s, n_0/s)) : (u, p^e s) = 1\}\right| \\
&= \sum_{e=0}^{t} \phi(p^{min(e,t-e)}) \sum_{s|n_0} |\{u \ (mod \ (s, n_0/s)) : (u, s) = 1\}| \\
&= \left(\sum_{e=0}^{t} \phi(p^{min(e,t-e)})\right) \sigma_\infty(n_0) \\
&= \left(p^{\lfloor \frac{t}{2} \rfloor} + p^{\lfloor \frac{t-1}{2} \rfloor}\right) \sigma_\infty(n_0)
\end{aligned}$$

Therefore, if $n = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$, then by the same argument as above we get

$$\sigma_\infty(n) = \prod_{k=1}^{r} \left(p_k^{\lfloor \frac{t_k}{2} \rfloor} + p_k^{\lfloor \frac{t_k-1}{2} \rfloor}\right). \tag{2.38}$$

Using the above equation and the multiplicativity of ϕ we get

$$\sigma_\infty(n) = \prod_{p^t \| n} \left(p^{\lfloor \frac{t}{2} \rfloor} + p^{\lfloor \frac{t-1}{2} \rfloor}\right) = \sum_{d|n} \phi\left((d, \frac{n}{d})\right), \tag{2.39}$$

for any $n \in \mathbb{N}$.

Remark 2.2 *Since $\Gamma^0(n) = T\Gamma_0(n)T^{-1} = T\Gamma_0(n)T$, $\Gamma^0(n)$ has $\sigma_\infty(n)$ inequivalent parabolic cusps and one can easily derive the analog of (iii) for $\Gamma^0(n)$.*

2.5 The Hecke Group

The group generated by translation $S_\lambda : \tau \to \tau + \lambda$ and inversion $T : \tau \to \frac{-1}{\tau}$, denoted by $H(\lambda)$, is called a Hecke group. Hecke [8] proved analytically that the group $H(\lambda) = \langle S_\lambda, T \rangle$ is discrete if and only if either $\lambda \geq 2$ or $\lambda = 2\cos(\frac{\pi}{q})$ for some $q \in \mathbb{Z}$ and $q \geq 3$. Later Evans [5] provided an elementary proof of this fact. From now on we restrict ourselves to discrete Hecke groups $H(\lambda)$ where $\lambda \in \mathcal{H} = \left\{2\cos\left(\frac{\pi}{q}\right) : q \in \mathbb{Z}, \text{ and } q \geq 3\right\}$

For $\lambda \in \mathcal{H}$ the group $H(\lambda)$ is generated by two elements of finite order

$$H(\lambda) = \langle T, \ P_\lambda \rangle, \tag{2.40}$$

where $P_\lambda : \tau \to \frac{-1}{\tau+\lambda}$. Since $T^2 = -I$, it is enough to show that P_λ has finite order. If $\theta = \frac{\pi}{q}$, then we can easily see that

$$P_\lambda = \begin{pmatrix} 0 & -1 \\ 1 & 2cos(\theta) \end{pmatrix} = \frac{1}{\sin\theta}\begin{pmatrix} -\sin(1-1)\theta & -\sin\theta \\ \sin\theta & \sin(1+1)\theta \end{pmatrix} \tag{2.41}$$

and

$$P_\lambda^2 = \frac{1}{\sin\theta}\begin{pmatrix} -\sin(2-1)\theta & -\sin 2\theta \\ \sin 2\theta & \sin(2+1)\theta \end{pmatrix}. \tag{2.42}$$

Similarly we can prove by induction that

$$P_\lambda^n = \frac{1}{\sin\theta}\begin{pmatrix} -\sin(n-1)\theta & -\sin n\theta \\ \sin n\theta & \sin(n+1)\theta \end{pmatrix}, \tag{2.43}$$

for any natural number n. Hence the smallest positive value of n for which $P_\lambda^n = I$ is $n = q$. This proves that the Hecke group $H(\lambda)$ is generated by two transformations of finite order.

2.5.1 Congruence Subgroups of $H(\lambda)$

The main results of this thesis require a discussion of some subgroups of the Hecke groups. The groups $H(\sqrt{2}) = H(2\cos(\frac{\pi}{4}))$ and $H(\sqrt{3}) = H(2\cos(\frac{\pi}{6}))$ are of particular interest since they are the only Hecke groups, aside from the modular group, whose elements are completely known.

It is well known([9], [30]) that $H(\sqrt{m})$, $m = 2, 3$, consists of all the mappings of the following two types:

(i) $M(\tau) = \frac{a\tau + b\sqrt{m}}{c\sqrt{m}\tau + d}$, $a, b, c, d \in \mathbb{Z}, ad - mbc = 1$,

(ii) $M(\tau) = \frac{a\sqrt{m}\tau + b}{c\tau + d\sqrt{m}}$, $a, b, c, d \in \mathbb{Z}, adm - bc = 1$.

Let $n \in \mathbb{N}$. Define

$$H_0^m(n) = \{M \in H(\sqrt{m}) : c \equiv 0 (\bmod n)\}. \quad (m = 2, 3)$$

Then $H_0^m(n)$ is a subgroup of $H(\sqrt{m})$. It is well known ([12]) that

(i) $$[H(\sqrt{m}) : H_0^m(n)] = \begin{cases} n\prod_{p|n}\left(1+\frac{1}{p}\right), & \text{if } (m,n) = 1 \\ 2n\prod_{p|n,p\neq m}\left(1+\frac{1}{p}\right), & \text{if } (m,n) = m \end{cases}$$

(ii) The number of inequivalent parabolic cusps of $H_0^m(n)$, denoted by $\sigma_\infty^m(n)$, is given by

$$\sigma_\infty^m(n) = \begin{cases} \sum_{d|n}\phi((d,n/d)), & \text{if } (m,n) = 1 \\ \sum_{d|n}\phi((d,mn/d)), & \text{if } (m,n) = m \end{cases} .$$

CHAPTER 3

FUNDAMENTAL DOMAIN AND HYPERBOLIC GEOMETRY

3.1 Definitions

Let Γ be a subgroup of $M\ddot{o}b(\mathbb{R})$. Then elements of Γ map the upper half-plane onto itself. Now let us define an equivalence relation on $\mathbb{H}_\infty$, which depends on Γ as follows:

$$\tau \overset{\Gamma}{\sim} \tau_0 \text{ iff } \exists M \in \Gamma \text{ such that } \tau = M(\tau_0) \ . \tag{3.1}$$

Clearly $\overset{\Gamma}{\sim}$ is an equivalence relation on $\mathbb{H}$; let the equivalence class determined by τ be denoted by $[\tau]$ and $\Gamma[\tau]$. Then

$$\Gamma[\tau] = [\tau] = \{M(\tau) : M \in \Gamma\}$$

and we call $\Gamma[\tau]$ the orbit of τ under Γ. An orbit $\Gamma[\tau]$ consists entirely of real points or entirely of nonreal points. For any two points τ and ω in $\mathbb{H}$ we can find a Möbius transformation $M \in M\ddot{o}b(\mathbb{R})$ such that $M(\tau) = \omega$. Hence $M\ddot{o}b(\mathbb{R})[\tau] = \mathbb{H}$. If Γ is a subgroup of $M\ddot{o}b(\mathbb{R})$, then by the Axiom of Choice there exists a set $\mathcal{F}$ such that $\mathcal{F}$ contains exactly one element from each equivalence class. Such sets are called fundamental sets. A singleton in $\mathbb{H}$ is a fundamental set for $M\ddot{o}b(\mathbb{R})$.

The references for the material in this chapter are [2], [11], [13],[15], [27] and [28].

Definition 3.1 *A set $\mathcal{F}$ of $\bar{\mathbb{H}}$ is called a fundamental set for the group Γ if it contains exactly one representatives of each equivalence class under Γ.*

If $\mathcal{D} \subset \mathcal{F}$, $M \in \Gamma$ and $\mathcal{F}$ is a fundamental set for Γ, then the set $M(\mathcal{D}) \cup (\mathcal{F} - \mathcal{D})$ is also a fundamental set for Γ. Therefore a fundamental set cannot be unique. A fundamental set $\mathcal{F}$ cannot be open, because if it is open then points on its boundary will not be equivalent to any of its elements and hence there exists an element $\tau \in \mathbb{H}$ such that $[\tau] \cap \mathcal{F} = \emptyset$. This is a contradiction. Since it is convenient to work with sets that are topologically "nice", at least open or closed, we shall modify the concept slightly and make the following definition.

Definition 3.2 *An open subset $\mathcal{R}$ of $\mathbb{H}$ is called a fundamental domain for Γ provided*

(1) no two distinct points of $\mathcal{R}$ are equivalent under Γ;

(2) every point of $\mathbb{H}$ is Γ-equivalent to a point of $\bar{\mathcal{R}}$.

Given a fundamental domain $\mathcal{R}$ for Γ we can always construct a fundamental set $\mathcal{F}$ for Γ such that $\mathcal{R} \subset \mathcal{F} \subset \bar{\mathcal{R}}$. Moreover, if a subgroup Γ of $M\ddot{o}b(\mathbb{R})$ has a nonempty fundamental domain $\mathcal{R}$ then the group Γ must be discrete. Suppose there exists a sequence $\{M_n\}$ of distinct elements of Γ such that $\lim_{n\to\infty} M_n = I$ and $M_n \neq I$. If $\tau \in \mathcal{R}$, then there exist a positive number δ such that $D(\tau, \delta) \subset \mathcal{R}$. Since $\lim_{n\to\infty} M_n = I$, there exists $N \in \mathbb{N}$ such that $M_n(\tau) \in D(\tau, \delta)$ for all $n \geq N$. Therefore the open set $\Omega = M_N(D(\tau, \delta)) \cap D(\tau, \delta) \subset \mathcal{R}$ contains $M_N(\tau)$. That means there exist a point $\omega \in \Omega$ different from the fixed points of M_N. Then there exists $\tau_1 \in D(\tau, \delta)$ such that $\omega = M_N(\tau_1)$. That means $\mathcal{R}$ contains two equivalent points ω and τ_1. This is impossible. Therefore Γ is discrete. The converse, which can be stated as: if a group is discrete then it has a fundamental domain, was proved by Poincaré [23]. Therefore we restrict ourselves to discrete subgroups of $M\ddot{o}b(\mathbb{R})$, and in particular to subgroups of the Hecke group $H(\lambda)$, $\lambda \in \mathcal{H}$.

3.2 Hyperbolic Geometry

We consider here Poincaré's model of the hyperbolic plane. A point in the hyperbolic plane is represented by a point in $\mathbb{H}$. A line is represented by the intersection of a vertical line or a circle orthogonal to $\mathbb{R}$ and the upper half-plane $\mathbb{H}$. We call these elements the h-plane, h-point, and h-line, respectively. In Euclidean geometry, there is one and only one line passing through any pair of distinct points. One can easily show that between any pair p, and q of distinct points of $\mathbb{H}$ there exists a unique h-line through p, and q. As in the Euclidean case, we say two h-lines are parallel if they are disjoint. We also define the angle measure in hyperbolic geometry to be the same as the angle measure in Euclidean geometry. Parallelism behaves much differently in hyperbolic geometry. Unlike the Euclidean case, if p is a point in $\mathbb{H}$ which is not on h-line $\mathcal{L}$ in $\mathbb{H}$, then there exist infinitely many different h-lines through p that are parallel to $\mathcal{L}$, as shown below. This is actually the main difference between these two geometries.

We can make $\mathbb{H}$ a metric space by defining a metric $d_{\mathbb{H}}$ as follows: let τ and ω be two distinct points in $\mathbb{H}$ and let $[\tau, \omega]$ be the h-line segment joining τ and ω. We can parametrize $[\tau, \omega]$ by a smooth map $\gamma : [0,1] \to [\tau, \omega]$. We define

$$d_{\mathbb{H}}(\tau, \omega) = \int_\gamma \frac{|dz|}{\Im(z)} = \int_0^1 \frac{|\gamma'(t)|}{\Im(\gamma(t))} dt, \tag{3.2}$$

and

$$d_{\mathbb{H}}(\tau, \tau) = 0. \tag{3.3}$$

We can see that $(\mathbb{H}, d_{\mathbb{H}})$ is a metric space and distance is preserved by Möbius

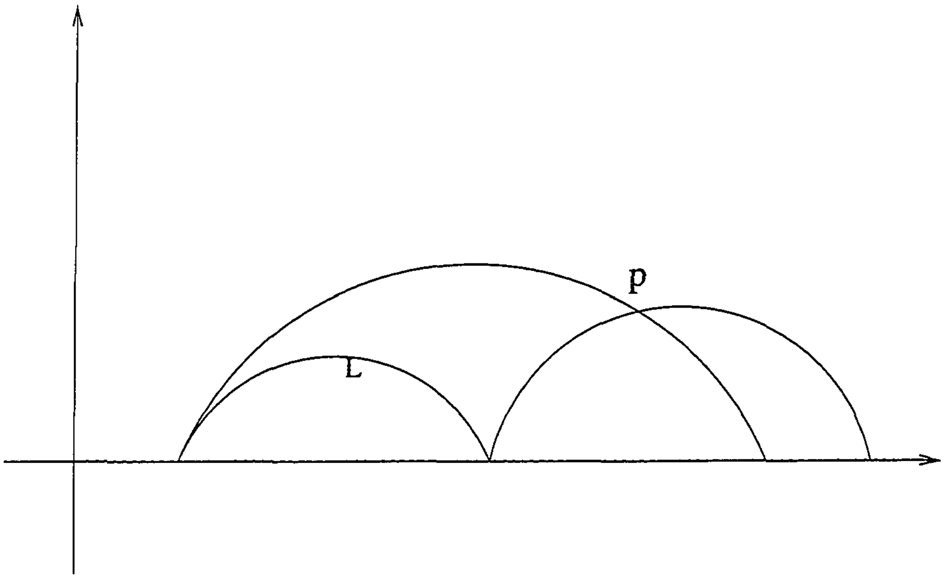


Figure 3.1: The two lines through P are || to L

transformations which fix $\mathbb{H}$; i.e., if $M \in Möb(\mathbb{R})$, then

$$d_{\mathbb{H}}(\tau, \omega) = d_{\mathbb{H}}(M(\tau), M(\omega)).$$

The topology defined by the metric $d_{\mathbb{H}}$ is equivalent to the usual euclidean topology. That is, every open set in one topology contains an open set in the other topology. Because of the invariance of the hyperbolic metric under Möbius transformations, in order to evaluate $d_{\mathbb{H}}(\tau, \omega)$ it is enough to evaluate $d_{\mathbb{H}}(ia, ib)$ for $b > a$. From the definition of the metric we get

$$d_{\mathbb{H}}(ia, ib) = \ln\left(\frac{b}{a}\right), \tag{3.4}$$

where $b > a$. The following theorem is very useful for evaluating and deriving properties of the above metric, and we will use it in this thesis when we discuss the existence of a fundamental domain for any discrete subgroup of $Möb(\mathbb{R})$.

Theorem 3.1 *For τ, $\omega \in \mathbb{H}$,*

(i) $d_{\mathbb{H}}(\tau,\omega) = \ln\left(\dfrac{|\tau-\bar{\omega}|+|\tau-\omega|}{|\tau-\bar{\omega}|-|\tau-\omega|}\right)$,

(ii) $\cosh\left(d_{\mathbb{H}}(\tau,\omega)\right) = 1 + \dfrac{|\tau-\omega|^2}{2\Im(\tau)\Im(\omega)}$,

(iii) $\sinh\left(\frac{1}{2}d_{\mathbb{H}}(\tau,\omega)\right) = \dfrac{|\tau-\omega|}{2\left(\Im(\tau)\Im(\omega)\right)^{1/2}}$,

(iv) $\cosh\left(\frac{1}{2}d_{\mathbb{H}}(\tau,\omega)\right) = \dfrac{|\tau-\bar{\omega}|}{2\left(\Im(\tau)\Im(\omega)\right)^{1/2}}$.

In order to prove the existence of a fundamental domain for any discrete subgroup of $M\ddot{o}b(\mathbb{R})$ we need the following definition.

Definition 3.3 *A perpendicular bisector of the h-line segment* $[\tau,\omega]$ *is the unique h-line* $\mathfrak{L}$ *passing through the midpoint of* τ *and* ω *orthogonal to the h-line segment* $[\tau,\omega]$ *(see Figure 3.2).*

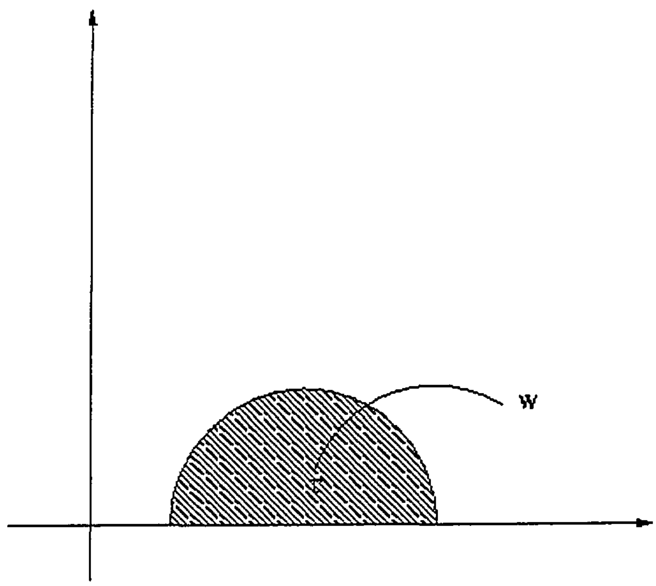


Figure 3.2:

Theorem 3.2 *A line given by the equation*

$$d_{\mathbb{H}}(\tau, \tau_0) = d_{\mathbb{H}}(\tau, \tau_1) \tag{3.5}$$

is the perpendicular bisector of the h-line segment $[\tau_0, \tau_1]$.

Proof: Without loss of generality we can assume that $\tau_0 = i$, and $\tau_1 = ib^2$ for some $b > 0$. By Theorem 3.1 the midpoint τ_2 of $[\tau_0, \tau_1]$ is given by $\tau_2 = ib$ and the perpendicular bisector is given by $|\tau| = b$. If $|\tau| = b$, then we can easily see that

$$\frac{|\tau - \tau_0|^2}{1} = \frac{|\tau - \tau_1|^2}{b^2}$$

and hence

$$1 + \frac{|\tau - \tau_0|^2/\Im(\tau_0)}{2\Im(\tau)} = 1 + \frac{|\tau - \tau_1|^2/\Im(\tau_1)}{2\Im(\tau)}.$$

By Theorem 3.1 we get

$$cosh(d_{\mathbb{H}}(\tau, \tau_0)) = cosh(d_{\mathbb{H}}(\tau, \tau_1)).$$

Therefore

$$d_{\mathbb{H}}(\tau, \tau_0) = d_{\mathbb{H}}(\tau, \tau_1).$$

■

Let $\mathfrak{L}$ be the perpendicular bisector of the h-line segment $[\tau_0, \omega_0]$. Then by Theorem 3.2 the equation of $\mathfrak{L}$ is given by $d_{\mathbb{H}}(\tau, \tau_0) = d_{\mathbb{H}}(\tau, \omega_0)$. The open set given by the equation $\{\tau \in \mathbb{H} : d_{\mathbb{H}}(\tau, \tau_0) < d_{\mathbb{H}}(\tau, \tau_1)\}$ is the open half-plane determined by $\mathfrak{L}$ and containing τ_0(the shaded part of Figure 3.2).

Notation 3.1 *We shall denote the perpendicular bisector of the h-line segment* $[\tau_0, M(\tau_0)]$ *by* $\mathfrak{L}_{\tau_0}(M)$ *and let* $H_{\tau_0}(M)$ *be the open half-plane determined by* $\mathfrak{L}_{\tau_0}(M)$ *and containing* τ_0*(similar to the shaded part of Figure 3.2).*

Next we define the notion of hyperbolic convexity of a set.

Definition 3.4 *A subset D of the hyperbolic plane is hyperbolically convex(h-convex) if for each pair of points τ and ω in D the closed h-line segment $[\tau, \omega]$ joining τ and ω is contained in D.*

Among the simplest examples of h-convex sets are $\mathbb{H}$, any h-line, and any closed h-line segment. As in Euclidean convexity, the intersection of a family of h-convex sets is h-convex. Given an h-line $\mathcal{L}$ its complement in the upper half-plane has two components, the two open half-planes determined by $\mathcal{L}$. We need some of the basic properties of h-convex sets for this thesis and we state them now. For the proofs of Theorems 3.3, 3.4 and 3.5 one can refer to the wonderful book of Beardon [2].

Theorem 3.3 *(1) If a set D is h-convex and $M \in Möb(\mathbb{R})$, then $M(D)$ is also h-convex.*

(2) Open half-planes are h-convex.

(3) The closure and interior of an h-convex set are h-convex.

(4) A subset D of the hyperbolic plane is h-convex if and only if D can be expressed as the intersection of a collection of half-planes.

Definition 3.5 *A hyperbolic polygon $\mathcal{P}$ is the interior of a closed Jordan curve*

$$[\tau_1, \tau_2] \cup [\tau_2, \tau_3] \cup \cdots \cup [\tau_{n-1}, \tau_n] \cup [\tau_n, \tau_1].$$

The interior angle θ_k of the polygon at τ_k is the angle determined by $D(\tau_k, r) \cap \mathcal{P}$ for all sufficiently small discs $D(\tau_k, r)$ centered at τ_k.

We allow the vertices τ_k to lie on $\bar{\mathbb{H}}$ and if $\tau_k \in \mathbb{R}_\infty$, then $\theta_k = 0$.

To prove the main theorem in this section we need the following theorem due to Tietze Heinrich [29]. We say that a set E is called locally convex if and only if for each point $\tau \in E$ there exists an open set U containing τ such that the set $E \cap U$ is convex. The notions of convexity and local convexity are meaningful in both Euclidean and hyperbolic spaces, and they extend in the obvious way to the closed hyperbolic plane.

Theorem 3.4 *Let P be the Euclidean plane or the closed hyperbolic plane. A closed subset E of P is convex if and only if it is connected and locally convex.*

Proof: If the result is true when P is the Euclidean plane, the relationship between the Poincaré and Beltrami-Klein models(the interested reader can refer to the book of Stahl [28] for an elementary introduction of these models) shows that the result is also true when P is a closed hyperbolic plane. Thus it is only necessary to show that if E is a closed, connected and locally convex subsets of $\mathbb{R}^2$ then E is convex (the reverse implication is trivial).

Note that any two points of E are connected by a polygonal curve in E. Because of this it is sufficient to show that if the Euclidean segments $[u, v]$, $[v, w]$ lie in E then so does the segment $[u, w]$. If u, v, w are collinear then this is trivial: thus we assume that these points are not collinear.

For each a, b, c, let $T(a, b, c)$ represent the closed triangle with vertices a, b, c (or the convex hull of the points a, b, c). Now let K be the set of x in (v, u) with the property that for some y in (v, w) we have $T(v, x, y) \subset E$. As E is locally convex at v, K contains some interval of positive length. Clearly, K is an interval of the form $[v, x)$ or $[v, x]$ where $x \neq v$ and we shall now show that $K = [v, u]$.

Choose a neighborhood N of x such that $E \cap N$ is convex and then choose $b \in [v, x) \cap N$ and $c \in [x, u] \cap N$ as shown in Figure 3.3. As $b \in K$, there exists some $y \in (v, w)$ with

$$T(v, b, y) \subset E.$$

Choose $z \in N \cap (b, y)$. As $E \cap N$ is convex we have

$$T(z, b, c) \subset E.$$

With a as shown in Figure 3.3 we also have

$$T(v, c, a) \subset T(v, b, y) \cup T(b, c, z) \subset E,$$

so $c \in K$. This shows that $x \in K$ and $x = u$ so $K = [v, u]$. Note that as $u \in K$, there is some $y \in (v, w)$ with $T(v, u, y) \subset E$.

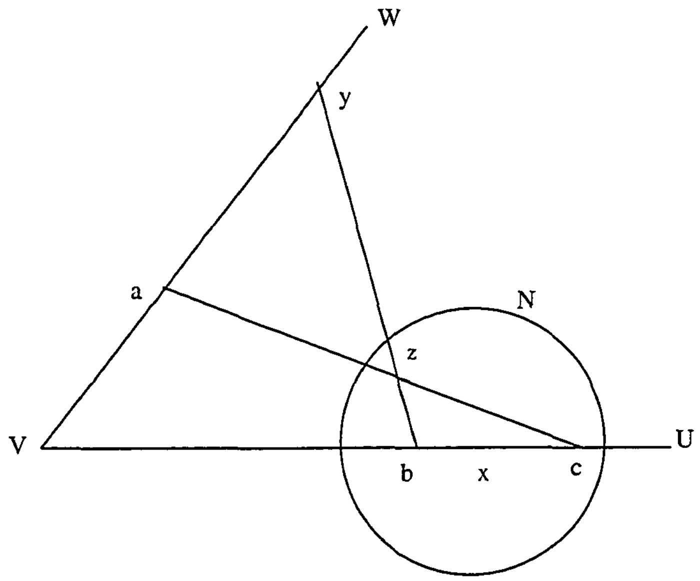


Figure 3.3:

Now consider the set K_1 of $y \in [v, w]$ such that $T(v, u, y) \subset E$. Exactly as before, K_1 is some segment $[v, y_0)$ or $[v, y_0]$. As E is closed, we see that $K_1 = [v, y_0]$. The argument in the preceding paragraph (with u, v, w replaced by u, y_0, w) shows that $y_0 = w$ so $w \in K_1$ and

$$T(v, u, w) \subset E.$$

■

Now we have all the necessary ingredients to prove one of the main results in hyperbolic geometry. This result is the main building block of this dissertation.

Theorem 3.5 *A hyperbolic polygon $\mathcal{P}$ with interior angles $\theta_1, \theta_2, \cdots, \theta_n$ is h-convex if and only if*

$$0 \leq \theta_k \leq \pi, \quad 1 \leq k \leq n$$

Proof: The conditions on the interior angles guarantees us that the set $\mathcal{P}$ is locally convex. Furthermore the set $\mathcal{P}$ is connected and closed. Therefore by Theorem 3.4 the polygon $\mathcal{P}$ is h-convex.

3.3 Existence of a fundamental domain for discrete subgroups of $M\ddot{o}b(\mathbb{R})$

In Section 3.1 we have seen that if a subgroup Γ of $M\ddot{o}b(\mathbb{R})$ has a nonempty fundamental domain, then the group must be discrete. In this section we will prove the converse.

Theorem 3.6 *Let Γ be a discrete subgroup of $M\ddot{o}b(\mathbb{R})$ and let τ_0 be a point in $\mathbb{H}$ which is not fixed by any nonidentity element of Γ. Then the set*

$$\mathcal{R} = \{\tau \in \mathbb{H} : d_{\mathbb{H}}(\tau, \tau_0) < d_{\mathbb{H}}(\tau, M(\tau_0)) \ \forall M \in \Gamma, \ \ M \neq I\} \tag{3.6}$$

is a fundamental domain for Γ. Moreover, $\mathcal{R}$ is h-convex. The set $\mathcal{R}$ is called a **Dirichlet Polygon** *for Γ centered at τ_0.*

Proof: We can easily see that

(i) $\mathcal{R} = \bigcap_{M\in\Gamma-\{I\}} H_{\tau_0}(M)$;

(ii) $\mathcal{R}$ is open.

Next we want to show that

(iii) $\overline{\mathcal{R}} = \bigcap_{M\in\Gamma-\{I\}} \overline{H_{\tau_0}(M)}$.

Since $\overline{\mathcal{R}} \subseteq \bigcap_{M\in\Gamma-\{I\}} \overline{H_{\tau_0}(M)}$, we have to show the reverse inclusion. For this we need the following fact: for any $\epsilon > 0$ and any $\tau \in \mathbb{H}$, the set $\Gamma[\tau] \cap D(\tau, \epsilon)$ contains only finitely many elements. First we want to show that for any $\epsilon > 0$, $\mathfrak{L}_{\tau_0}(M)$ intersects $D(\tau_0, \epsilon)$ for only finitely many $M \in \Gamma - \{I\}$. If $\mathfrak{L}_{\tau_0}(M) \cap D(\tau_0, \epsilon) \neq \emptyset$, then $d_{\mathbb{H}}(\omega, \tau_0) < \epsilon$ for some $\omega \in \mathfrak{L}_{\tau_0}(M)$. By definition of $\mathfrak{L}_{\tau_0}(M)$ we have $d_{\mathbb{H}}(M(\tau_0), \tau_0) \leq d_{\mathbb{H}}(\omega, \tau_0) + d_{\mathbb{H}}(M(\tau_0), \omega) < 2\epsilon$ and therefore $M(\tau_0) \in D(\tau_0, 2\epsilon)$. Using the fact that we mentioned earlier we can conclude that there are only finitely many $M \in \Gamma - \{I\}$ such that $\mathfrak{L}_{\tau_0}(M) \cap D(\tau_0, \epsilon) \neq \emptyset$. Next we will show that $\mathfrak{L}_{\tau_0}(M) \cap D(\tau, \epsilon) \neq \emptyset$ for only finitely many elements $M \in \Gamma - \{I\}$, for a given $\tau \in \mathbb{H}$ and $\epsilon > 0$. Let $\tau \in \mathbb{H}$

and $\epsilon > 0$. If $\delta = \epsilon + d_{\mathbb{H}}(\tau, \tau_0)$, then we can show that $D(\tau, \epsilon) \subset D(\tau_0, \delta)$. Since there are only finitely many $M \in \Gamma$ such that $\mathfrak{L}_{\tau_0}(M) \cap D(\tau_0, \delta) \neq \emptyset$, then there will be finitely many $M \in \Gamma$ such that $\mathfrak{L}_{\tau_0}(M) \cap D(\tau, \epsilon) \neq \emptyset$.

Now let $\tau \in \bigcap_{M \in \Gamma - \{I\}} \overline{H_{\tau_0}(M)}$. Then $\tau \in \overline{H_{\tau_0}(M)}$ $\forall M \in \Gamma - \{I\}$. If $\tau \in H_{\tau_0}(M)$ $\forall M \in \Gamma - \{I\}$, then $\tau \in \mathcal{R} \subseteq \bar{\mathcal{R}}$. Suppose that there exists $N \in \Gamma$ such that $\tau \in \overline{H_{\tau_0}(N)} \backslash H_{\tau_0}(N)$. That means $d_{\mathbb{H}}(\tau, \tau_0) = d_{\mathbb{H}}(\tau, N(\tau_0))$. We want to show that any deleted neighborhood of τ intersects $\mathcal{R}$ nontrivially. For any $\epsilon > 0$ there exist only a finite number of elements of Γ, say $M_1, \cdots, M_n \in \Gamma$ such that

$$\mathfrak{L}_{\tau_0}(M_k) \cap D(\tau, \epsilon) \neq \emptyset \quad \forall k = 1, \cdots, n.$$

Then there exist $\tau' \in \mathbb{H}$ such that

$$\tau' \in (D(\tau, \epsilon) \backslash \{\tau\}) \cap \bigcap_{k=1}^{n} H_{\tau_0}(M_k).$$

Moreover, since $\tau \in \overline{H_{\tau_0}(M)}$ $\forall M \in \Gamma - \{I\}$, we must have

$$\tau \in H_{\tau_0}(M) \ \ \forall M \in \Gamma - \{I, M_1, \cdots, M_n\}$$

and

$$d_{\mathbb{H}}(\tau, \tau_0) > \epsilon.$$

Therefore

$$D(\tau, \epsilon) \subset H_{\tau_0}(M) \ \ \forall M \in \Gamma \backslash \{I, M_1, \cdots, M_n\}$$

and hence $\tau' \in \mathcal{R} \cap D(\tau, \epsilon)$. Thus $\tau \in \overline{\mathcal{R}}$. Therefore

$$\overline{\mathcal{R}} = \bigcap_{M \in \Gamma - \{I\}} \overline{H_{\tau_0}(M)}.$$

To complete the proof we need to show that $\overline{\mathcal{R}}$ contains a point from each equivalence class and $\mathcal{R}$ does not contain two equivalent points. Let $\tau \in \mathbb{H}$. Since Γ is a discrete group, the Γ-orbit $\Gamma[\tau]$ is a discrete set and hence there exists $\tau^* \in \Gamma[\tau]$ with the smallest hyperbolic distance from τ_0. Then

$$d_{\mathbb{H}}(\tau_0, \tau^*) \leq d_{\mathbb{H}}(\tau_0, M(\tau^*)) \ \ \forall M \in \Gamma \backslash \{I\}$$

and hence $\tau^* \in \overline{H_{\tau_0}(M)}$ for all $M \in \Gamma - \{I\}$. Thus $\tau^* \in \overline{\mathcal{R}}$, so $\overline{\mathcal{R}}$ contains at least one point from every Γ-orbit. It remains to show that no two distinct points $\tau,\ \omega \in \mathcal{R}$ can lie in the same Γ-orbit. Let $\tau,\ \omega \in \mathcal{R}$ and distinct. Then

$$d_{\mathbb{H}}(\tau_0, \tau) < d_{\mathbb{H}}(\tau_0, M(\tau)),\ \forall M \in \Gamma - \{I\}$$

and

$$d_{\mathbb{H}}(\tau_0, \omega) < d_{\mathbb{H}}(\tau_0, M(\omega)),\ \forall M \in \Gamma - \{I\}.$$

If τ and ω lie in the same Γ-orbit, then there exists $N \in \Gamma - \{I\}$ such that $\omega = N(\tau)$ and

$$d_{\mathbb{H}}(\tau_0, \tau) < d_{\mathbb{H}}(\tau_0, N(\tau)) = d_{\mathbb{H}}(\tau_0, \omega)$$

and

$$d_{\mathbb{H}}(\tau_0, \omega) < d_{\mathbb{H}}(\tau_0, N^{-1}(\omega)) = d_{\mathbb{H}}(\tau_0, \tau).$$

This is a contradiction. Therefore $\mathcal{R}$ contains at most one point from each Γ-orbit. Clearly $\mathcal{R}$ is open and h-convex. Therefore $\mathcal{R}$ is a path-connected fundamental domain for Γ.■

The above theorem is not well adapted for the actual construction of an h-convex fundamental domain. Among other reasons, it may be very difficult to determine the orbit $\Gamma[\tau_0]$ of τ_0, and so to determine the points of $\mathbb{H}$ closer to τ_0 than to any other point in the orbit. We shall describe another method, associated with the name of L.R.Ford, which is based on the *isometric circle*. Let Γ be a discrete subgroup of $M\ddot{o}b(\mathbb{R})$ containing translations. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ with $c \neq 0$. Then the circle

$$I(M) := \{\tau \in \mathbb{H} : |c\tau + d| \ = \ 1\} \tag{3.7}$$

is called the **isometric circle of** M. The map M acts like a euclidean rigid motion on the circle $I(M)$, and this explains the name. It is easy to see that M maps $I(M)$ on $I(M^{-1})$ and maps $Ext\,(I(M))$ on $Int\,(I(M^{-1}))$. Since Γ contains a translation, there exists a smallest positive number β such that

$S_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \Gamma$. Let

$$\mathbb{S}_\beta = \left\{ \tau \in \mathbb{H} : \frac{-\beta}{2} < \Re(\tau) < \frac{\beta}{2} \right\}. \tag{3.8}$$

Note that if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c = 0$, then $M \in \Gamma_\infty$. Therefore every $M \in \Gamma - \Gamma_\infty$ has an isometric circle. Furthermore, if we have a sequence $\{I(M_n)\}$ of distinct isometric cirles, where $M_n \in \Gamma$, then the radii of the isometric circles $I(M_n)$ tend to zero. Ford [6] was the first to exploit this property of isometric circles to prove the following theorem.

Theorem 3.7 *The set $\mathcal{F}$ defined by*

$$\mathcal{F} = \mathbb{S}_\beta \cap \left\{ \bigcap_{M \in \Gamma - \langle S_\beta \rangle} Ext\,(I(M)) \right\} \tag{3.9}$$

is a h-convex fundamental domain for Γ. $\mathcal{F}$ is called Ford fundamental domain.

Proof: Suppose $\tau \in \mathcal{F}$ and $M \in \Gamma$. If $M \in \Gamma_\infty$, then M translates τ out of $\mathbb{S}_\beta$ hence out of $\mathcal{F}$. Otherwise τ lies outside $I(M)$; therefore $M(\tau)$ lies inside $I(M^{-1})$ and so outside $\mathcal{F}$. Distinct points of $\mathcal{F}$ are not equivalent under Γ.

We must now show that every $\omega \in \mathbb{H}$ is equivalent to a point of $\overline{\mathcal{F}}$. The radii of the isometric circles are bounded above, as we have observed. Hence there is a $\alpha > 0$ such that $\tau = x + iy \in \overline{\mathcal{F}}$ if $\tau \in \overline{\mathbb{S}}_\beta$ and $y > \alpha$.

Let $\tau \in \mathbb{H}$ be a boundary point of $\mathcal{F}$. Then τ may lie on one of the vertical sides bounding $\mathbb{S}_\beta$. If not, τ lies on some isometric circle, for certainly τ does not lie inside an isometric circle. Now τ does not lie on infinitely many isometric circles, for the radii of these circles tend to zero and this would force τ to be a point of $\mathbb{R}$. A boundary point of $\mathcal{F}$ lies on a finite number of isometric circles but inside none. It follows that a point of $\mathbb{H}$ not in $\overline{\mathcal{F}}$ must lie inside some isometric circle.

Now let $\tau_0 \in \mathbb{H}$. Translate τ to a point $\tau_1 \in \overline{\mathbb{S}}_\beta$ by an element of Γ, writing

$$\tau_0 = x_0 + iy_0, \qquad \tau_1 = x_1 + iy_1,$$

where $y_1 = y_0$. If τ_1 is not in $\overline{\mathcal{F}}$, it is inside some isometric circle $I(M_1)$, $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, and we have

$$\tau_2 = x_2 + iy_2 = M_1(\tau_1) \qquad y_2 = \frac{y_1}{|c_1\tau_1 + d_1|^2} > y_1.$$

Next, translate τ_2 to a point in $\overline{\mathbb{S}}_\beta$, and so on. We obtain a sequence

$$\tau_0,\ \tau_1,\ \cdots,\ \tau_n,\ \cdots$$

with

$$y_0 = y_1 < y_2 = y_3 < y_4 = y_5 < \cdots.$$

If for some n, $y_{2n+1} \geq \alpha$, then $\tau_{2n+1} \in \overline{\mathcal{F}}$ and the proof ends. Otherwise there is an infinite sequence $\tau_1 = M_1(\tau_0)$, $\tau_3 = M_3(\tau_0)$; $\tau_5 = M_5(\tau)$, $\cdots$, all images of τ_0 under Γ and all lying in $\overline{\mathbb{S}}_\beta$. This sequence has a point of accumulation in $\tau^* = x^* + iy^* \in \overline{\mathbb{S}}_\beta$ and $y^* > 0$, since y_{2n+1} is strictly increasing. Therefore

$$Im(M_n(\tau_0)) = \frac{y_0}{(c_n x_0 + d_n)^2 + (c_n y_0)^2} \to y^*$$

and

$$|M_n(\tau_0)|^2 = \frac{(a_n x_0 + b_n)^2 + (a_n y_0)^2}{(c_n x_0 + d_n)^2 + (c_n y_0)^2} \to |\tau^*|^2$$

Hence $\{c_n\}$, $\{d_n\}$, $\{a_n\}$ and $\{b_n\}$ are bounded. That means the sequence $\{M_n\}$ of distinct elements of Γ has a convergent subsequence. This is a contradiction.

Even though the above theorem is important, it is not well adapted for the actual construction of an h-convex fundamental domain. Among other reasons, it may be very difficult to get the isometric circles and hence the intersection of their exterior, because the result is highly dependent on knowing almost all of the elements of the group. This necessitates the development of a new and practical method. Here we shall describe another method which is very convenient in constructing fundamental domains for subgroups Γ of a group G having a fundamental domain with nice topological properties.

Theorem 3.8 *Let Γ be the a discrete subgroup of $M\ddot{o}b(\mathbb{R})$ and Γ_2 be a subgroup of Γ of index n. If*

$$\Gamma \;=\; \Gamma_2 A_1 \cup \Gamma_2 A_2 \cup \cdots \cup \Gamma_2 A_n \tag{3.10}$$

is a decomposition of Γ into Γ_2-right cosets and if $\mathcal{R}$ is a fundamental domain of Γ, then the set

$$\mathcal{R}_2 \;= \bigcup_{k=1}^{n} A_k\,(\mathcal{R}) \tag{3.11}$$

is a fundamental domain for Γ_2.

Proof: We first prove that no two distinct points of $\mathcal{R}_2$ are equivalent under the group Γ_2. Towards this end suppose that τ_1 and τ_2 are in $\mathcal{R}_2$ and equivalent under Γ_2. Then there exist $\omega_1,\ \omega_2 \in \mathcal{R},\ i,\ j \in \{1, 2, 3, \cdots, n\}$, and $M \in \Gamma_2$ such that

$$\tau_2 = M(\tau_1),\ \tau_1 = A_i(\omega_1),\ \tau_2 = A_j(\omega_2).$$

It follows that $A_j(\omega_2) = MA_i(\omega_1)$ or $\omega_2 = A_j^{-1}MA_i(\omega_1)$. But $A_j^{-1}MA_i \in \Gamma$; hence $\omega_1 = \omega_2$ since $\mathcal{R}$ is a fundamental domain for Γ. Hence $A_j^{-1}MA_i = I$ or $A_jA_i^{-1} = M \in \Gamma_2$. This implies that $i = j$. Hence $\tau_1 = \tau_2$, so that no two distinct points of $\mathcal{R}_2$ are equivalent under Γ_2.

Now we want to show that any point of $\mathbb{H}$ is equivalent under Γ_2 to a point in $\overline{\mathcal{R}_2}$. Given any $\tau \in \mathbb{H}$, there exists $N \in \Gamma$ such that $\tau = N(\omega)$, where $\omega \in \overline{\mathcal{R}}$. On the other hand there exist $M \in \Gamma_2$ and $i (1 \le i \le n)$ such that $N = MA_i$. Hence $\tau = N(\omega) = MA_i(\omega) = M(\omega_1)$, where $\omega_1 = A_i(\omega) \in A_i(\overline{\mathcal{R}}) = \overline{A_i(\mathcal{R})}$. Therefore $\omega_1 \in \overline{\mathcal{R}_2}$. ■

Definition 3.6 *Let $\mathcal{R}$ be a fundamental domain for Γ. A parabolic point (or a parabolic vertex, or a parabolic cusp, or a cusp) of Γ in $\mathcal{R}$ is any point $q \in \mathbb{R}_\infty$ such that $q \in \overline{\mathcal{R}}$.*

3.4 Fundamental Domain for $\Gamma(1)$ and $H(\lambda)$ and Their subgroups.

Notation:

- $\mathcal{R}_0 := \{\tau \in \mathbb{H} : |\Re(\tau)| < \frac{1}{2},\ |\tau| > 1\}$;
- $\mathcal{R}^0 := \{\tau \in \mathbb{H} : 0 < \Re(\tau) < \frac{1}{2},\ |\tau - 1| > 1\}$;
- $\mathcal{R}_\lambda := \{\tau \in \mathbb{H} : |\Re(\tau)| < \frac{\lambda}{2},\ |\tau| > 1\}$;
- $\mathcal{R}^\lambda := \{\tau \in \mathbb{H} : 0 < \Re(\tau) < \frac{\lambda}{2},\ |\tau - \frac{1}{\lambda}| > 1\}$;
- $\mathcal{S}_\delta := \{\tau \in \mathbb{H} : |\Re(\tau)| < \frac{1}{2},\ |\Im(\tau)| \geq \delta\}$ where $\delta > 0$.

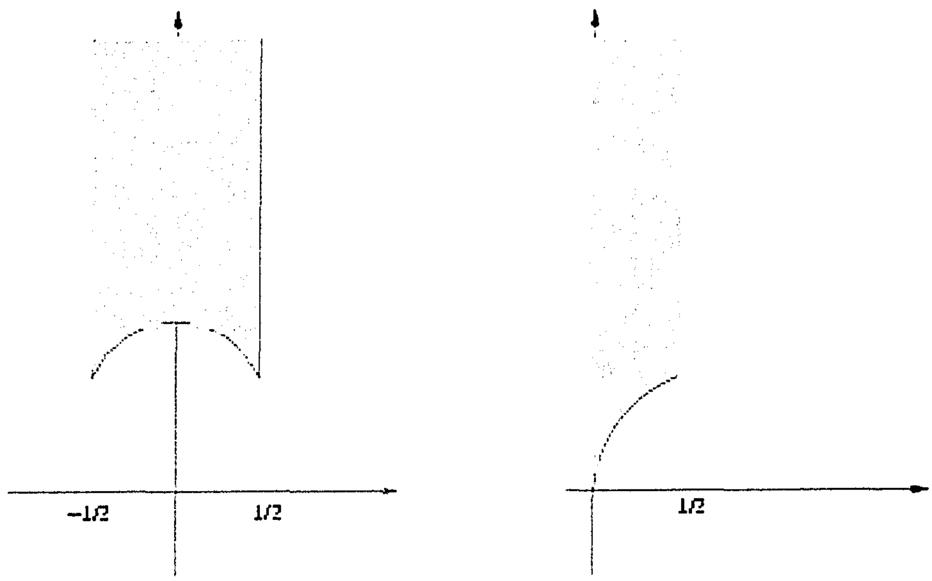


Figure 3.4: $\mathcal{R}_0$ and $\mathcal{R}^0$

In this section we will show that the sets $\mathcal{R}_0$, and $\mathcal{R}^0$ are fundamental domains for $\Gamma(1)$ and the sets $\mathcal{R}_\lambda$, and $\mathcal{R}^\lambda$ are fundamental domains for $H(\lambda)$.

Theorem 3.9 *The sets $\mathcal{R}_0$ and $\mathcal{R}^0$ are fundamental domains for the full modular group $\Gamma(1)$.*

Before we prove the above theorem let us prove the following lemma which is useful for the proof of the theorem.

Lemma 3.1 *For any $\delta > 0$ and $\tau \in \mathbb{H}$ the set $\Gamma(1)[\tau] \cap \mathcal{S}_\delta$ is a finite set.*

Proof:

Suppose that $\tau = x + iy \in \mathbb{H}$, $\delta > 0$ and c, d be a pair of integers such that $(c, d) = 1$, $c \neq 0$. Then there exists a unique $M_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ such that $\left|\frac{a}{c}\right| \leq \frac{1}{2}$. If $N \in \Gamma(1)$ and $N = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1)$, then there exists a unique $n \in \mathbb{Z}$ such that $N = S^n M_{c,d}$. Now let $N(\tau) \in \mathcal{S}_\delta$ and $N = S^n M_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

Now,
$$\begin{aligned} &\Im(N(\tau)) \geq \delta \\ \Leftrightarrow\ &\frac{y}{(cx+d)^2 + c^2y^2} \geq \delta \\ \Leftrightarrow\ &(cx+d)^2 + c^2y^2 \leq \tfrac{y}{\delta} \\ \Leftrightarrow\ &(x^2+y^2)c^2 + (2x)cd + d^2 \leq \tfrac{y}{\delta}. \end{aligned}$$

But the last inequality has finitely many integer solutions c, d with $(c, d) = 1$. For each solution c, d with $(c, d) = 1$, $c \neq 0$, there exists a unique n such that $\Im(S^n M_{c,d}(\tau)) \geq \delta$ and $|\Re(S^n M_{c,d}(\tau))| \leq \frac{1}{2}$. If $c = 0$, then the number of possibilities for d is 2 and hence there are only finitely many elements $N \in \Gamma(1)$ such that $N = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and $N(\tau) \in \mathcal{S}_\delta$. Therefore $\Gamma(1)[\tau] \cap \mathcal{S}_\delta$ is a finite set.■

Proof of Theorem 3.9

First we shall show that for any $\tau \in \mathbb{H}$ there exists $\omega \in \mathcal{R}_0$ such that $\tau \in \Gamma(1)[\omega]$. Let $\tau = x + iy \in \mathbb{H}$. We call $y = \Im(\tau)$ the height of τ. There exists

$n_1 \in \mathbb{Z}$ such that if $\tau_1 := S^{n_1}(\tau)$, then $|\Re(\tau_1)| \leq \frac{1}{2}$ and $\Im(\tau_1) = \Im(\tau)$. If $\tau_1 \notin \bar{\mathcal{R}}_0$, then $|\tau_1| < 1$. Clearly $T(\tau_1) \in [\tau]$ and we note that

$$\Im(T(\tau_1)) = \frac{\Im(\tau_1)}{|\tau_1|^2} > \Im(\tau_1) = \Im\tau.$$

Further, there exists $n_2 \in \mathbb{Z}$ such that if $\tau_2 := S^{n_2}(T(\tau_1)) = S^{n_2}TS^{n_1}(\tau)$, then $|\Re(\tau_2)| \leq \frac{1}{2}$ and

$$\Im(\tau_2) = \Im(S^{n_2}T(\tau_1)) = \Im(T(\tau_1)) > \Im(\tau_1) = \Im\tau = y.$$

Either $\tau_2 \in \overline{\mathcal{R}}_0$ or else we can continue the process and find a congruent point τ_3 with $|\Re(\tau_3)| \leq \frac{1}{2}$ and $\Im(\tau_3) > \Im(\tau_2)$. By the above lemma $\mathcal{S}_y \cap [\tau]$ is finite and hence the process ultimately terminates after a finite number of steps, say after k steps. Then the point $\tau_k \in [\tau] \cap \bar{\mathcal{R}}_0$.

Now it remains to show that each orbit contains at most one point of $\mathcal{R}_0$. Suppose that $\tau = x + iy$ and $\omega = u + iv$ are two distinct points in $\mathcal{R}_0$ which are equivalent under $\Gamma(1)$. Assume, with out loss of generality, that $y \geq v$ and $\tau = M(\omega)$, where $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1)$. This implies

$$y = \Im(M(\omega)) = \frac{v}{|c\omega + d|^2}.$$

Hence $|c\omega + d| \leq 1$. We cannot have $|c| \geq 2$, since no circle of radius $r \leq \frac{1}{2}$ and orthogonal to $\mathbb{R}$ intersects $\mathcal{R}_0$. If $c = 0$, then $M = S^n$, for some $n \in \mathbb{Z}$, which is clearly impossible. We may therefore assume that $c = 1$. Then d is either -1, 0 or 1. One can check that all of the above cases are impossible. That means we cannot find a map $M \in \Gamma(1)$ such that $\tau = M(\omega)$. Therefore each orbit contains at most one point of $\mathcal{R}_0$. Thus, $\mathcal{R}_0$ is a fundamental domain for $\Gamma(1)$. To show that $\mathcal{R}^0$ is also a fundamental domain for $\Gamma(1)$, first note that $\mathcal{R}_0 = A \cup B$, where

$$A := \left\{\tau \in \mathbb{H} : -\frac{1}{2} < \Re(\tau) \leq 0,\ |\tau| > 1\right\},$$

and

$$B := \left\{\tau \in \mathbb{H} : \frac{1}{2} > \Re(\tau) > 0,\ |\tau| > 1\right\}.$$

Then $\mathcal{R}^0 = \left(\overline{B \cup T(A)}\right)^o$ and by the remark we made in section 3.1 $\mathcal{R}^0$ is a fundamental domain for $\Gamma(1)$. ∎

The following theorem was first proved by Hecke [8] and later by Evans [5] using elementary arguments as in the above theorem. Here we state the theorem without proof.

Theorem 3.10 *The sets $\mathcal{R}_\lambda$ and $\mathcal{R}^\lambda$ are fundamental domains for the Hecke groups $H(\lambda)$ when $\lambda \geq 2$ or $\lambda \in \mathcal{H}$.*

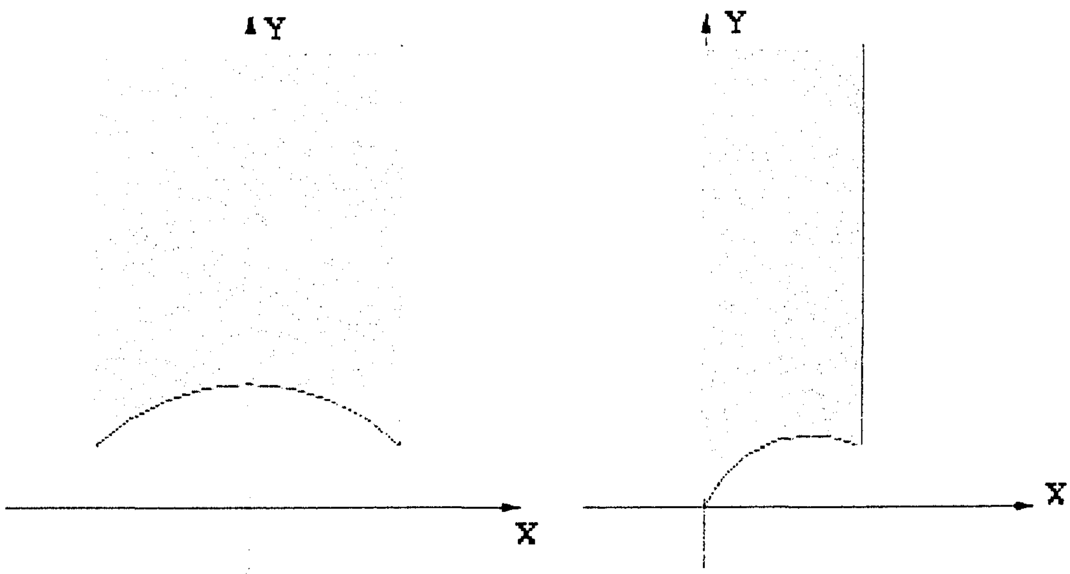


Figure 3.5: $\mathcal{R}_\lambda$ and $\mathcal{R}^\lambda$

For convenience we shall combine Theorems 2.8 and 3.8 in the following theorem.

Theorem 3.11 *Let Γ be a subgroup of $\Gamma(1)$ of index μ. Then*

(1) Let $\Sigma = \{A_1, \cdots, A_\mu\}$ be a complete right coset system of Γ in $\Gamma(1)$. Then

$$\Gamma(1) = \Gamma \cdot A_1 \cup \Gamma \cdot A_2 \cup \cdots \cup \Gamma \cdot A_\mu = \bigcup_{A \in \Sigma} \Gamma \cdot A \tag{3.12}$$

and the set

$$\mathcal{R}(\Gamma) = \bigcup_{k=1}^{\mu} A_k(\mathcal{R}_0) \tag{3.13}$$

is a fundamental domain for Γ*;*

(2) there exists a finite number of elements $A_1, \cdots, A_p$ *such that*

$$\Gamma(1) = \bigcup_{k=1}^{p} \bigcup_{j=0}^{\lambda_k - 1} \Gamma A_k S^j \tag{3.14}$$

and the set

$$\mathcal{R}_\Gamma = \bigcup_{k=1}^{p} \bigcup_{j=0}^{\lambda_k - 1} A_k S^j(\mathcal{R}_0) \tag{3.15}$$

is a fundamental domain for Γ*, where* λ_k *is as defined in Theorem 2.8 for* $N = S$*.*

Definition 3.7 *The set* $\mathcal{R}(\Gamma)$ *constructed as in Theorem 3.11 is called a standard fundamental domain(SFD) for* Γ *and the set* $\mathcal{R}_\Gamma$ *is called a cuspidal standard fundamental domain(CSFD) for* Γ*. The positive integer* λ_k *is called the width of the parabolic cusp* $A_k(i\infty)$*.*

Remark 3.1 *(i) The parabolic cusps of* Γ *in the SFD* $\mathcal{R}(\Gamma)$ *are the points given by* $A_k(i\infty)$ *for* $k = 1, 2, \cdots, \mu$*. In this case we may have two distinct parabolic cusps that are* Γ*-equivalent to each other.*

(ii) The parabolic cusps of Γ *in the CSFD* $\mathcal{R}_\Gamma$ *are the points given by* $A_k(i\infty)$ *for* $k = 1, 2, \cdots, p$*. In this case no two distinct parabolic cusps are* Γ*-equivalent to each other, and this justifies the name cuspidal standard fundamental domain.*

(iii) If two cusps are Γ*-equivalent, then they have the same width.*

(iv) The number p *in part(2) of the above theorem is the number of inequivalent parabolic cusps or the parabolic class number and we usually denote it by* $\sigma_\infty(\Gamma)$*.*

(v) $\mathcal{R}(\Gamma)$ and $\mathcal{R}_\Gamma$ depend on the complete right coset system.

Now we are ready to state the questions raised by my advisor, Professor Marvin I.Knopp, and by Professor Mark Sheingorn, which have lead to this dissertation. First, is the set $\mathcal{R}(\Gamma)$ connected? If not is it possible to choose a complete right coset system $\{A_1, \cdots, A_\mu\}$ to make $\mathcal{R}(\Gamma)$ connected? Second, is the set $\mathcal{R}(\Gamma)$ h-convex? If not is it possible to choose a complete right coset system $\{A_1, \cdots, A_\mu\}$ to make $\mathcal{R}(\Gamma)$ h-convex? Third, is the set $\mathcal{R}_\Gamma$ h-convex? If not is it possible to choose a reduced right coset system $\{A_1, \cdots, A_{\sigma_\infty(\Gamma)}\}$ to make $\mathcal{R}_\Gamma$ h-convex? We will address these questions in the next two chapters. (Note that the two fundamental domains constructed by Ford and Poincaré are h-convex.)

CHAPTER 4

CONSTRUCTION OF A CONNECTED CSFD AND ITS LIMITATIONS

In the first section we will give an algorithmic construction of a connected cuspidal standard fundamental domain of Γ, where $[\Gamma(1) : \Gamma] < \infty$, which can be implemented on a computer using any programming language. Actually I have written a Maple package which gives me such domains for the congruence subgroup $\Gamma_0(n)$ for any $n \in \mathbb{N}$. The reference for the material of the first section is [27]. In Section 4.3 we give an example of a subgroup Γ of $\Gamma(1)$ of finite index which has no h-convex cuspidal standard fundamental domain. This result appears to be new.

4.1 Construction of Connected CSFD

We need the following lemmas in order to give an algorithmic proof of the construction of a connected cuspidal standard fundamental domain. We call the image of $\mathcal{R}_0$ by any member of a modular group *a modular triangle.*

Lemma 4.1 *1. There are exactly 6 modular triangles attached at the elliptic point* $\rho = e^{\frac{2\pi i}{3}}$;

2. there are exactly 6 modular triangles attached at $\zeta \in [\rho]$;

3. there are exactly two modular triangles attached at each $\zeta \in [\omega]$, *where* $\omega \in \partial(\mathcal{R}_0) - \{\rho, \rho+1, \infty\}$.

Proof: A modular triangle is said to be *attached* at the point ω, if ω is on its boundary. Usually ω will be an elliptic point.

Consideration of angles would be the ideal way of proving the claim. However, we give here an elementary algebraic proof. Suppose that $M(\tau) = \frac{a\tau+b}{c\tau+d}$ with $ad - bc = 1$ and $\rho \in M(\overline{\mathcal{R}_0})$. Then either $\rho = M(\rho)$ or $\rho = M(\rho+1)$.

$$\begin{aligned} M(\rho) = \rho \quad &\Leftrightarrow a\rho + b = c\rho^2 + d\rho \\ &\Leftrightarrow (a-d)\rho + b - c\rho^2 = 0 \\ &\Leftrightarrow (a-d)\rho + b + c(\rho+1) = 0 \\ &\Leftrightarrow (a-d+c)\rho + b + c = 0 \\ &\Leftrightarrow (a+c-d) = 0, \text{ and } b + c = 0 \\ &\Leftrightarrow M = I, \text{ or } M = TS, \text{ or } M = S^{-1}T. \end{aligned}$$

If $M(\rho+1) = \rho$, then $MS(\rho) = \rho$ and by the first part $MS = I$ or $MS = TS$ or $MS = S^{-1}T$ and hence $M = T$ or $M = S^{-1}$ or $M = TST$. Therefore there are exactly six modular triangles attached at the elliptic point ρ.

The proofs of (2) and (3) are similar and hence left to the reader.

Two modular triangles are said to be *adjacent* if they have a common side.

Applying the definition of a fundamental domain for a subgroup of the modular group, we get the following lemma.

Lemma 4.2 *(a) If* M_1, $M_2 \in \Gamma$ *with* $M_1 \neq M_2$ *and* $\mathcal{R}$ *is a subset of a fundamental domain, then* $M_1(\mathcal{R}) \cap M_2(\mathcal{R}) = \emptyset$.

(b) If $A(i\infty) \overset{\Gamma}{\nsim} B(i\infty)$ *and* $A(\mathcal{R}_0)$ *is adjacent to* $B(\mathcal{R}_0)$, *then* $A = BT$.

Theorem 4.1 *For any subgroup* Γ *of* $\Gamma(1)$ *with* $[\Gamma(1):\Gamma] = \mu < \infty$ *we can choose a reduced right coset system* $\{A_1, A_2, \cdots, A_p\}$ *suitably to make the CSFD* $\overline{\mathcal{R}_\Gamma}$ *connected.*

Proof: If $\mu = 1$, then $\Sigma = \{I\}$, *and* $\mathcal{R} = \mathcal{R}_0$, which is connected. Therefore we may assume that $\mu \geq 2$.

STEP 1. Pick any element A_1 of $\Gamma(1) - \Gamma$ and form the $\mathfrak{S}_1 := \{A_1, A_1 S, \cdots, A_1 S^{\lambda_1 - 1}\}$, where λ_1 is the smallest positive integer such that $A_1 S^{\lambda_1} A_1^{-1}$ belongs to Γ. Clearly $\mathcal{R}_1 = \overline{\bigcup_{L \in \mathfrak{S}_1} L(\mathcal{R}_0)}$ is connected.

Terminate the process if either all modular triangles which are adjacent to $\mathcal{R}_1$ are equivalent to one of the modular triangles contained in $\mathcal{R}_1$ or $|\mathfrak{S}_1| = \mu$. (Later we will show that these two statements are equivalent.) Otherwise go to the next step.

STEP 2. There exists $A_2 \in \Gamma(1)$ such that $A_2 \overset{\Gamma}{\not\sim} \mathfrak{S}_1$ and $A_2(\mathcal{R}_0)$ is adjacent to $M(\mathcal{R}_0)$ for some $M \in \mathfrak{S}_1$. If λ_2 is the smallest positive integer such that $A_2 S^{\lambda_2} A_2^{-1} \in \Gamma$, then form $\mathfrak{S}_2 = \mathfrak{S}_1 \cup \{A_2, A_2 S, \cdots, A_2 S^{\lambda_2 - 1}\}$. Because of the choice of A_2, $\mathcal{R}_2 = \overline{\bigcup_{L \in \mathfrak{S}_2} L(\mathcal{R}_0)}$ is connected.

Terminate the process if either all modular triangles which are adjacent to $\mathcal{R}_2$ are equivalent to one of the modular triangles contained in $\mathcal{R}_2$ or $|\mathfrak{S}_2| = \mu$. (Later we will show that these two statements are equivalent.) Otherwise go to the next step.

STEP k. Suppose we get $\mathfrak{S}_{k-1} = \bigcup_{j=1}^{k-1} \{A_j, A_j S, \cdots, A_j S^{\lambda_j - 1}\}$ and $\mathcal{R}_{k-1} = \overline{\bigcup_{L \in \mathfrak{S}_{k-1}} L(\mathcal{R}_0)}$ from STEP k-1 such that $\mathcal{R}_{k-1}$ is connected and there exists $A_k \in \Gamma(1) - \Gamma \cdot \mathfrak{S}_{k-1}$ with $A_k(\mathcal{R}_0)$ adjacent to $\mathcal{R}_{k-1}$ and $A_k \overset{\Gamma}{\not\sim} \mathfrak{S}_{k-1}$. If λ_k is the smallest positive integer such that $A_2 S^{\lambda_2} A_2^{-1} \in \Gamma$, then form $\mathfrak{S}_k := \mathfrak{S}_{k-1} \cup \{A_k, A_k S, \cdots, A_k S^{\lambda_k - 1}\}$. Because of the choice of A_k, $\mathcal{R}_k = \overline{\bigcup_{L \in \mathfrak{S}_k} L(\mathcal{R}_0)}$ is connected.

Since $[\Gamma(1) : \Gamma] = \mu < \infty$, the process terminates after a finite number of steps, say at STEP p. Note that the process terminates if either $|\mathfrak{S}_p| = \mu$, in which case there is nothing to do, or all the modular triangles that are adjacent to $\mathcal{R}_p$ are equivalent to one of the modular triangles contained in $\mathcal{R}_p$.

We want to show that these statements are equivalent. Suppose that the latter one is true. That is, if $M(\mathcal{R}_0)$ is adjacent to $\mathcal{R}_p$, then $M \overset{\Gamma}{\sim} \mathfrak{S}_p$ or equivalently $M \in \Gamma \cdot \mathfrak{S}_p$. Since $\mathcal{R}_p$ has a finite number of sides, $M(\mathcal{R}_p)$, $M \in \Gamma$, has a finite number of sides and at each side of $M(\mathcal{R}_p)$ there exists a modular triangle which is adjacent to it. Moreover, if a modular triangle $M_1(\mathcal{R}_0)$ is adjacent to $M(\mathcal{R}_p)$ for some $M \in \Gamma$, then $M^{-1}M_1(\mathcal{R}_0)$ is adjacent to $\mathcal{R}_p$. Therefore $M^{-1}M_1 \in \Gamma \cdot \mathfrak{S}_p$ and since $M \in \Gamma$ we can conclude that $M_1 \in \Gamma \cdot \mathfrak{S}_p$.

Now we define two sets as follows:

$$A = \bigcup_{M \in \Gamma \cdot \mathfrak{S}_p} M(\mathcal{R}_0),$$

$$B = \bigcup_{M \in \Gamma(1) - \Gamma \cdot \mathfrak{S}_p} M(\mathcal{R}_0).$$

Let $\zeta \in \bigcup_{L \in \Gamma(1)} L(\partial(\mathcal{R}_0) - \{\infty\})$. We want to show that if some modular triangle attached at ζ belongs to A, then all modular triangles attached at ζ belong to A. Consequently, $\zeta \in (\overline{A})^o$. First note that

$$A = \bigcup_{M \in \Gamma} M(\mathcal{R}_p)$$

and there are at most six modular triangles attached at ζ. Suppose that a modular triangle $M(\mathcal{R}_0)$ attached at ζ belongs to A. Then there exist $M_1 \in \Gamma$, and $L_1 \in \mathfrak{S}_p$ such that $M = M_1 L_1$. Since $M(\overline{\mathcal{R}_0}) \subseteq M_1(\overline{\mathcal{R}_p})$, $\zeta \in M_1(\overline{\mathcal{R}_p})$.

Let $\zeta \in [\rho]$. In this case we have to show that all the six copies of $\mathcal{R}_0$ attached at ζ belong to A. If $\zeta \in (M_1(\overline{\mathcal{R}_p}))^o$, then all the six copies of $\mathcal{R}_0$ attached at ζ belong to $M_1(\mathcal{R}_p)$ and hence belong to A. Otherwise $\zeta \in \partial M_1(\overline{\mathcal{R}_p})$.

Then at the two sides of $M_1(\mathcal{R}_p)$ containing ζ there exist $M_2, M_3 \in \Gamma$ such that $M_1(\mathcal{R}_p)$ is adjacent to $M_2(\mathcal{R}_p)$ on one side and adjacent to $M_3(\mathcal{R}_p)$ on the other side. Now either all the six modular triangles attached at ζ belong to $M_1(\mathcal{R}_p) \cup M_2(\mathcal{R}_p) \cup M_3(\mathcal{R}_p)$ or there exist at most three modular triangles attached at ζ not contained in $M_1(\mathcal{R}_p) \cup M_2(\mathcal{R}_p) \cup M_3(\mathcal{R}_p)$. If the former is true, then we are done. Otherwise one of the remaining modular triangles $N(\mathcal{R}_0)$ attached at ζ which are not contained in $M_1(\mathcal{R}_p) \cup M_2(\mathcal{R}_p) \cup M_3(\mathcal{R}_p)$ must be adjacent to either $M_2(\mathcal{R}_p)$ or $M_3(\mathcal{R}_p)$. Without loss of generality we may assume that $N(\mathcal{R}_0)$ is adjacent to $M_2(\mathcal{R}_p)$. There exists $M_4 \in \Gamma$ such that $M_4(\mathcal{R}_p)$ contains at least $N(\mathcal{R}_0)$ and $M_4(\mathcal{R}_p)$ is adjacent to $M_2(\mathcal{R}_p)$. Hence A contains at least 4 modular triangles attached at ζ. Continuing this argument we can conclude that all the six modular triangles attached at ζ belong to A. Therefore $\zeta \in (\overline{A})^o$.

Let $\zeta \in L(\partial(\mathcal{R}_0) - \{\rho, \rho+1, \infty\})$. In this case there are only two modular triangles attached at ζ. Now we want to show that both of them belong to A. If $\zeta \in (M_1(\overline{\mathcal{R}_p}))^0$, then both copies of $\mathcal{R}_0$ attached at ζ belong to $M_1(\mathcal{R}_p)$ and hence belong to A. Now let $\zeta \in \partial M_1(\overline{\mathcal{R}})$. Then there exists $M_2 \in \Gamma$ such that $M_1(\mathcal{R}_p)$ is adjacent to $M_2(\mathcal{R}_p)$. Therefore A contains both copies of $\mathcal{R}_0$ attached at ζ. Hence $\zeta \in (\overline{A})^o$.

Therefore if some modular triangle attached at ζ belongs to A, then $\zeta \in (\overline{A})^o$.

If $\zeta \in \mathbb{H}$, then $\zeta \in M(\bar{\mathcal{R}}_0)$ for some $M \in \Gamma(1)$. If there exists $M_1 \in \Gamma \cdot \mathfrak{S}_p$ such that $\zeta \in M_1(\mathcal{R}_0)$, then $\zeta \in \bar{A}^o$. Otherwise all the modular triangles containing ζ belong to B and hence $\zeta \in \bar{B}^o$. Therefore $\zeta \in (\overline{A})^o$ or $\zeta \in (\overline{B})^o$. From the definition of the two sets we can see that $\bar{A}^o \cap \bar{B}^o = \emptyset$. So far we have shown

(i) $\mathbb{H} = (\overline{A})^o \cup (\overline{B})^o$;

(ii) $(\overline{A})^o \cap (\overline{B})^o = \emptyset$;

(iii) $(\overline{A})^o \cap \mathbb{H} \neq \emptyset$.

Since $\mathbb{H}$ is connected and $(\overline{A})^o \cap \mathbb{H} \neq \emptyset$, then $B = \emptyset$. That means $\Gamma(1) - \Gamma \cdot \mathfrak{S}_p = \emptyset$. Hence $|\mathfrak{S}_p| = \mu$ and $(\overline{\mathcal{R}_p})^o$ is a connected cuspidal standard fundamental domain for Γ.

q.e.d

Remark 4.1 *The above Theorem is also true if we replace the full modular group by the Hecke group $H(\lambda)$. The only thing which we need to modify in order to carry out the proof of the theorem is Lemma 3.1: at the elliptic point of the Hecke group $H(\lambda)$, where $\lambda = 2cos(\frac{\pi}{q})$, there are at most $2q$ replicas of the triangle $\mathcal{R}_\lambda$ attached.*

We get the following corollary from the proof of the above Theorem.

Corollary 4.1 *Let $[\Gamma(1) : \Gamma] = \mu < \infty$ and suppose Σ is a finite set consisting of inequivalent elements of $\Gamma(1)$ modulo Γ. If $\mathcal{R} = \bigcup_{A \in \Sigma} \overline{A(\mathcal{R}_0)}$ is connected and every modular triangle adjacent to $\mathcal{R}$ is equivalent to a modular triangle contained in $\mathcal{R}$, then $|\Sigma| = \mu$.*

4.2 Examples

Example 4.1 We will show that the group $\Gamma^0(p)$, where p is prime, has an h-convex CSFD. Using the above Theorem we get a right coset decomposition

$$\Gamma(1) = \Gamma^0(p) \cdot \{S^{-1}, I, \cdots, S^{p-2}, T\}.$$

Let

$$\mathcal{R} = \bigcup_{j=-1}^{p-2} S^j(\mathcal{R}_0) \cup T(\mathcal{R}_0). \tag{4.1}$$

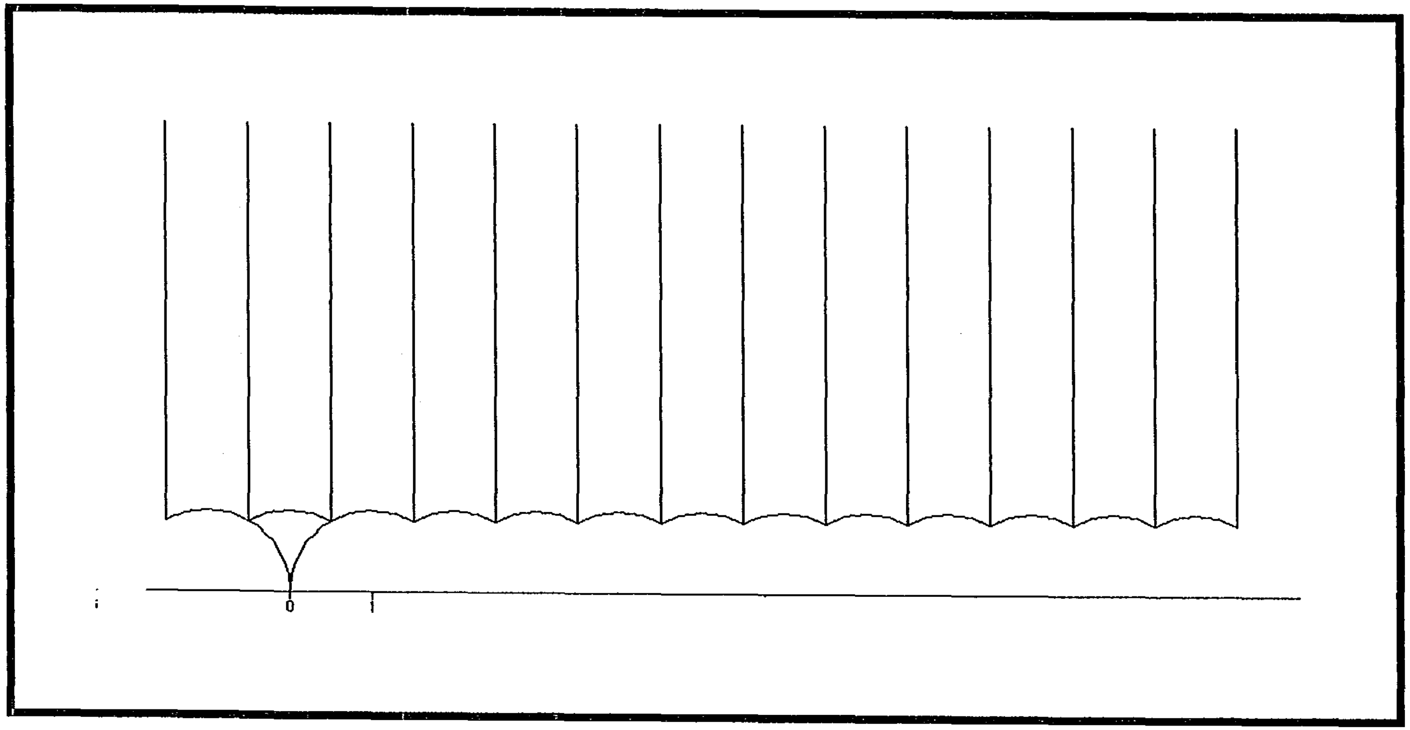


Figure 4.1: H-convex SFD for $\Gamma^0(13)$

Then we can easily show that $(\bar{\mathcal{R}})^o$ is h-convex CSFD. Figure 4.1 shows $\mathcal{R}$ for $p = 13$.

The sides of the CSFD $\mathcal{R}$ given in equation 4.1 are paired by the transformations $TS^{-1}T$, S^p, and S^kTS^{-m}, where $k = 2, 3, \cdots, p-2$ and m is the solution of the equation $km = -1(\text{modp})$ in the interval $-2 \leq m \leq p-2$. Therefore by the Poincaré's Theorem [23, p. 255] the group $\Gamma^0(p)$ is generated by a subset of these transformations, and this is consistent with Rademacher's result [24].

Example 4.2 We will show that the group $\Gamma^0(p^2)$, where p is prime, has an h-convex CSFD. Applying Theorem 2.15 to the given group we get

(i) $[\Gamma(1) : \Gamma^0(p^2)] = p^2 + p$

(ii) $\sigma^0_\infty(p^2) = p + 1$

(iii)

k	Cusp $q_k \in$	width λ_k
1	$[\infty]$	p^2
2	$[0]$	1
3	$[p]$	1
..	..	..
p+1	$[p(p-1)]$	1

Now we want to choose $p+1$ transformations $A_1, \cdots, A_{p+1}$ such that

$$\Gamma(1) = \bigcup_{k=1}^{p+1} \Gamma^0(p^2) \cdot \{A_k,\ A_kS,\ \cdots,\ A_kS^{\lambda_k - 1}\}$$

and

$$\mathcal{R} = \bigcup_{k=1}^{p+1} \bigcup_{j=0}^{\lambda_k - 1} A_kS^j(\mathcal{R}_0)$$

is an h-convex CSFD. Let $A_1 = S^{-1}$, $A_2 = T$, $A_3 = S^pT$, ..., $A_{p+1} = S^{p(p-1)}T$. Then we can easily see that

$$\Gamma(1) = \bigcup_{k=1}^{p+1} \Gamma^0(p^2) \cdot \{A_k,\ A_kS,\ \cdots,\ A_kS^{\lambda_k - 1}\}$$

and the set

$$\mathcal{R} = \bigcup_{k=1}^{p+1} \bigcup_{j=0}^{\lambda_k-1} A_k S^j(\mathcal{R}_0)$$

is a cuspidal standard fundamental domain for $\Gamma^0(p^2)$. It remains to show that $\bar{\mathcal{R}}$ is h-convex. First note that the only vertices of $\bar{\mathcal{R}}$ in $\mathbb{H}$ are

$$\rho - 1, \ \rho, \ \ \rho + 1, \ \rho + 2, \ \cdots, \ \ \rho + p^2 - 2, \ \ \text{and} \ \ \rho + \mathrm{p}^2 - 2$$

By Theorem 3.5 it is enough to show that the $\mathcal{R}$ is locally h-convex at those vertices listed above. The interior angle at the vertices

$$\rho, \ \rho + 1, \ \rho + p - 1, \ ,\rho + p \ \cdots, \ \ \rho + p(p-1) - 1, \ \ \text{and} \ \ \rho + \mathrm{p}(\mathrm{p}-1)$$

equals $3\left(\frac{\pi}{3}\right) = \pi$ and the interior angle at the remaining vertices except $\rho - 1$ and $\rho + p^2 - 2$ equals $\frac{2\pi}{3}$ and at those exceptional vertices the interior angle equals $\frac{\pi}{3}$. Therefore $\overline{\mathcal{R}}$ is locally h-convex. Since $\overline{\mathcal{R}}$ is closed, connected and locally h-convex, $\overline{\mathcal{R}}$ is h-convex. Figure 4.2 shows an H-convex SFD for $\Gamma^0(25)$.

The sides of the CSFD described above are paired by the transformations $TS^{-1}T$, S^{p^2}, , $S^{lp}TSTS^{-lp}$, where $l = 0, 1, \cdots, p-1$ and S^tTS^{-k} where $2 \leq k \leq p^2-2$ and $p \nmid k \pm 1$ and t is a solution of the equation $km = -1(\mathrm{mod} p)$ in the interval $-2 \leq m \leq p-2$. Therefore by the Poincaré's Theorem [23, p. 255] the group $\Gamma^0(p^2)$ is generated by a subset these transformations.

Example 4.3 We will show that the group $\Gamma^0(pq)$, where p and q are distinct primes, has an h-convex CSFD. We can easily see that $\sigma_\infty(pq) = 4$. Since $(p, q) = 1$, there exist integers l and r such that $0 \leq l \leq q-1$ and $pl - qr = 1$. If we choose $A_1 = S^{-pl}$, $A_2 = S^{-pl}T$, $A_3 = T$, and $A_4 = S^{q(p-r)}TS^{1-p}$, then

$$\Gamma(1) = \Gamma_0(pq) \cdot \bigcup_{k=1}^{4} \{A_k, A_kS, \cdots, A_kS^{\lambda_k-1}\} \tag{4.2}$$

where $\lambda_1 = pq, \ \ \lambda_2 = q, \ \ \lambda_3 = 1, \ \ \lambda_4 = p.$

Therefore the set given by

$$\mathcal{R} = \bigcup_{k=1}^{4} \bigcup_{j=0}^{\lambda_k-1} A_k S^j(\mathcal{R}_0)$$

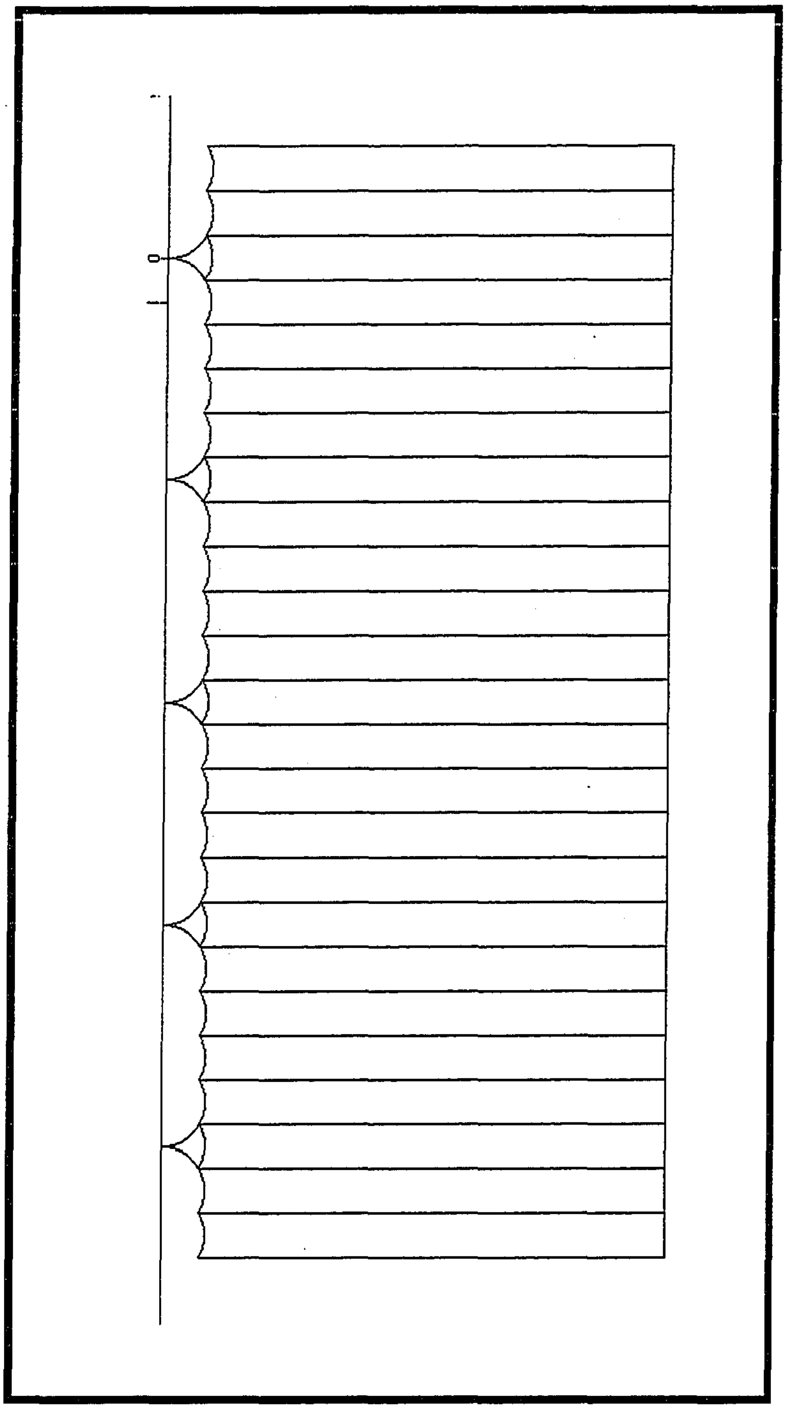


Figure 4.2: H-convex SFD for $\Gamma^0(25)$

is a CSFD for $\Gamma^0(pq)$. Furthermore we can show that $\overline{\mathcal{R}}$ is h-convex. Figure 4.3 shows an h-convex CSFD for $\Gamma^0(15)$.

To provide the generators of the group $\Gamma^0(pq)$ by applying the Poincaré's Theorem we need to know the sides of the domain found above. The sides of the domain given above are:

TYPE I:	$S^k(\mathcal{C}) = S^kT(\mathcal{C});\quad k = -pl+1, ..., -2, 2, ..., q(p-r)-1,$
TYPE II:	$S^{-pl}T(\mathcal{L}),\ S^{-pl}TS^q(\mathcal{L}),\ S^{q(p-r)}TS(\mathcal{L}),\ S^{q(p-r)}TS^{1-p}(\mathcal{L}),$
TYPE III:	$S^{-pl}TS^n(\mathcal{C}) = S^{-pl}TS^nT(\mathcal{C});\quad n = 2, 3, ..., q-1$ and
	$S^{q(p-r)}TS^{-m}(\mathcal{C}) = S^{q(p-r)}TS^{-m}T(\mathcal{C});\quad m = 2, 3, ..., p-1,$
TYPE IV:	$S^{-pl}(\mathcal{L}) \cup S^{-pl}TS(\mathcal{C}),\ S^{q(p-r)}(\mathcal{L}) \cup S^{q(p-r)}TS^{-1}(\mathcal{C}),\ TS(\mathcal{L}) \cup S^{-1}(\mathcal{C}),$
	$T(\mathcal{L}) \cup S(\mathcal{C}).$

Sides of TYPE II are mapped to sides of the same category and as a result we get two side-pairing transformations

$$M_1 = S^{pl}TS^qTS^{pl} \text{ and } \mathrm{M}_2 = \mathrm{S}^{\mathrm{q(p-r)}}\mathrm{TS}^{-\mathrm{p}}\mathrm{TS}^{-\mathrm{q(p-r)}}.$$

Sides of TYPE IV are mapped to sides of the same category and as a result we get two side-pairing transformations

$$M_3 = S^{pq} \text{ and } \mathrm{M}_4 = \mathrm{TST}.$$

Sides of TYPE III are mapped to sides of TYPE I. For each $n = 2, 3, ..., q-1$ there exists an $s \in \{1-l, ..., q-l-1\}$ such that $s(n-1)+l \equiv 0(\mathrm{mod} q)$. Hence the side-pairing transformation

$$N_n = S^{-pl}TS^nTS^{-ps}$$

pairs the sides $S^{ps}(\mathcal{C})$ and $S^{-pl}TS^n(\mathcal{C})$. For each $t = 2, 3, ..., p-1$ there exists an $m \in \{-r+1, ..., p-r-1\}$ such that $m(1-t)+r \equiv 0$ (modp). Hence the transformation

$$B_t = S^{q(p-r)}TS^{-t}TS^{-qm}$$

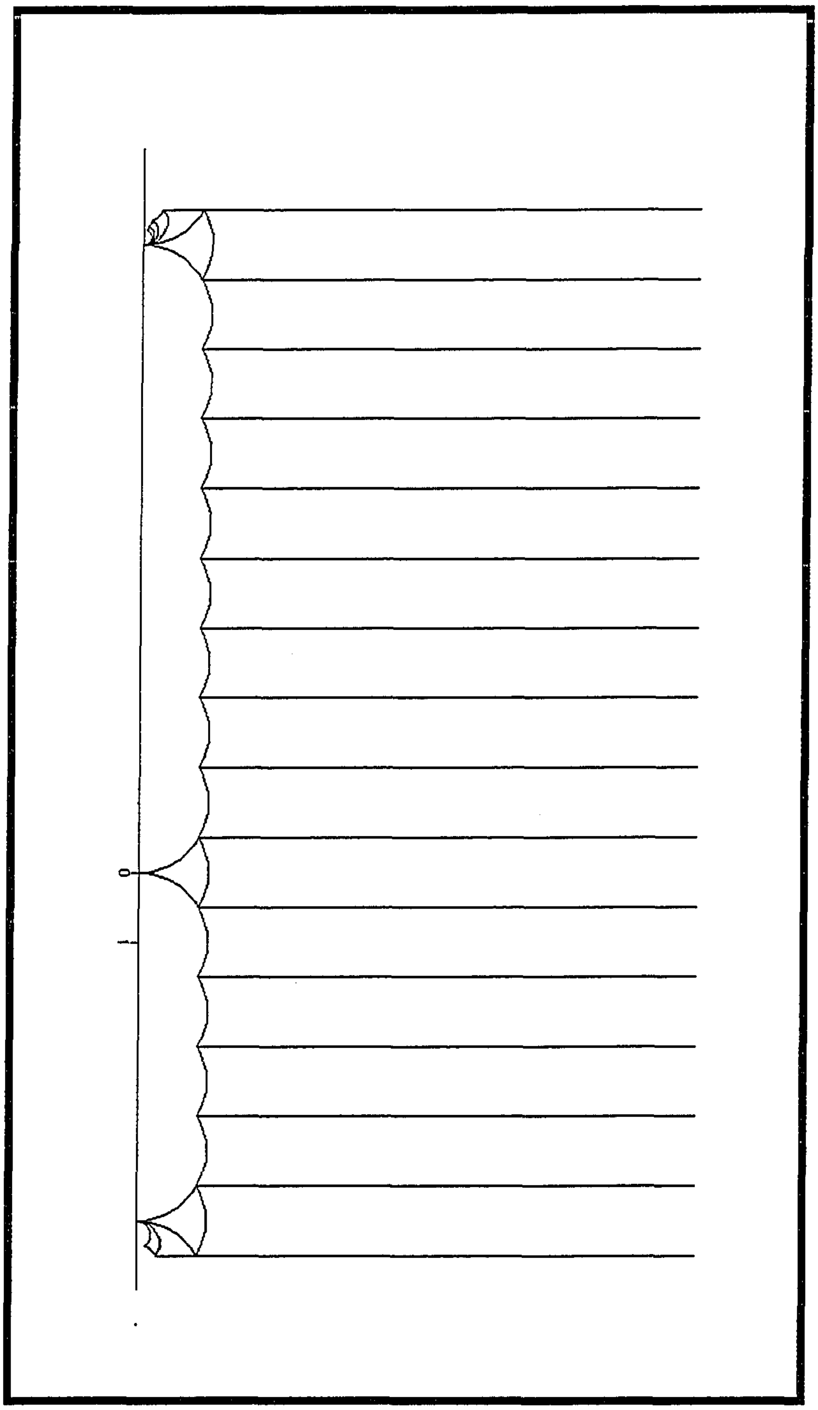


Figure 4.3: H-convex SFD for $\Gamma^0(15)$

pairs the sides $S^{qm}(\mathcal{C})$ and $S^{q(p-r)}TS^{-t}(\mathcal{C})$.

All the remaining sides of TYPE I are mapped to sides of the same category. Let $\mathfrak{A} = \{k \in \mathbb{Z} : 1 - pl \leq k \leq q(p-r) - 1, |k| > 1, (k,p) = (k,q) = 1\}$. For each $k \in \mathfrak{A}$, there exists $h \in \mathfrak{A}$ such that $k(pq - h) \equiv 1 \pmod{pq}$. Hence the transformation

$$U_k = S^h T S^{-k}$$

pairs the sides $S^k(\mathcal{C})$ and $S^h T(\mathcal{C})$.

Therefore the transformations

$$M_j(1 \leq j \leq 4),\ N_n(2 \leq n \leq q-1),\ B_t(2 \leq t \leq p-1), \text{and} \mathrm{U_k}(\mathrm{k} \in \mathfrak{A})$$

generate the group $\Gamma^0(pq)$.

Can we continue this forever and show that any subgroup of finite index of the modular group has an h-convex cuspidal standard fundamental domain? The answer is no and the following section gives an example of a subgroup without such a fundamental domain.

4.3 Limitations

Proposition 4.1 *The subgroup* $\Gamma^0(125)$ *of* $\Gamma(1)$ *defined by*

$$\Gamma^0(125) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : 125 | b \right\}$$

has no h-convex cuspidal standard fundamental domain.

Proof: Applying Theorem 2.15 to the group $\Gamma^0(125)$ we get:

1. $[\Gamma(1) : \Gamma^0(125)] = 150.$

2. The number of inequivalent cusps of $\Gamma^0(125)$ is $\sigma_\infty(125) = 10.$

3.

k	Cusp $q_k \in$	width λ_k
1	$[\infty]$	125
2	[0]	1
3	[25]	1
4	[50]	1
5	[75]	1
6	[100]	1
7	[5]	5
8	[10]	5
9	[15]	5
10	[20]	5

4. If two reduced rationals $\frac{a_1}{c_1}$ and $\frac{a_2}{c_2}$ are equivalent modulo $\Gamma^0(125)$, then $(a_1, 125) = (a_2, 125)$.

We prove the proposition by contradiction. Suppose there are 10-transformations $A_1, A_2, \dots, A_{10}$ such that

$$\Gamma(1) = \Gamma^0(125). \bigcup_{k=1}^{10} \{A_k, A_kS, ..., A_kS^{\lambda_k - 1}\}$$

and if $\mathcal{R}_k = \bigcup_{j=0}^{\lambda_k-1} A_kS^j(\mathcal{R}_0)$ for $k = 1, 2, \cdots, 10$ and $\mathcal{R} = \cup_{k=1}^{10}\mathcal{R}_k$, then $\mathcal{R}$ is CSFD for $\Gamma^0(125)$ such that $\overline{\mathcal{R}}$ is h-convex.

Without loss of generality we can assume that the first cusp to be ∞. Since $S^{125} \in \Gamma^0(125)$ we can, without loss of generality, take $A_1 = S^{-l}$ for $l \in \mathbb{Z}$ and $0 \leq l \leq 125 - 1$. The h-convex polygon $\overline{\mathcal{R}_1}$ looks like the shaded region in Figure 4.4.

Suppose that $A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ for $k = 1, 2, \cdots, 10$. By definition $\bar{\mathcal{R}_k}$ is an h-convex polygon for all k. Moreover, since $\bar{\mathcal{R}}$ is h-convex and hence connected, $\mathcal{R}_2$ must share a side with $\mathcal{R}_k$ for some $k \in \{1, 3, 4, ..., 10\}$. We want to show that $k = 1$. Since $\mathcal{R}_2$ shares a side with $\mathcal{R}_k$, there exists $0 \leq \alpha_k < \lambda_k$ such that $A_2(\mathcal{R}_0)$ shares a side with $A_kS^{\alpha_k}(\mathcal{R}_0)$. By Lemma 4.2 $A_2 = A_kS^{\alpha_k}T = \begin{pmatrix} b_k + \alpha_k a_k & -a_k \\ d_k + \alpha_k c_k & -c_k \end{pmatrix}$. From the discussion above $125|a_2$ and hence $125|(b_k + \alpha_k a_k)$. If $k \neq 1$, then $5|a_k$ and hence $5|b_k$. This is a contradiction. Therefore $k = 1$ and $\mathcal{R}_2$ shares a side with $\mathcal{R}_1$ only.

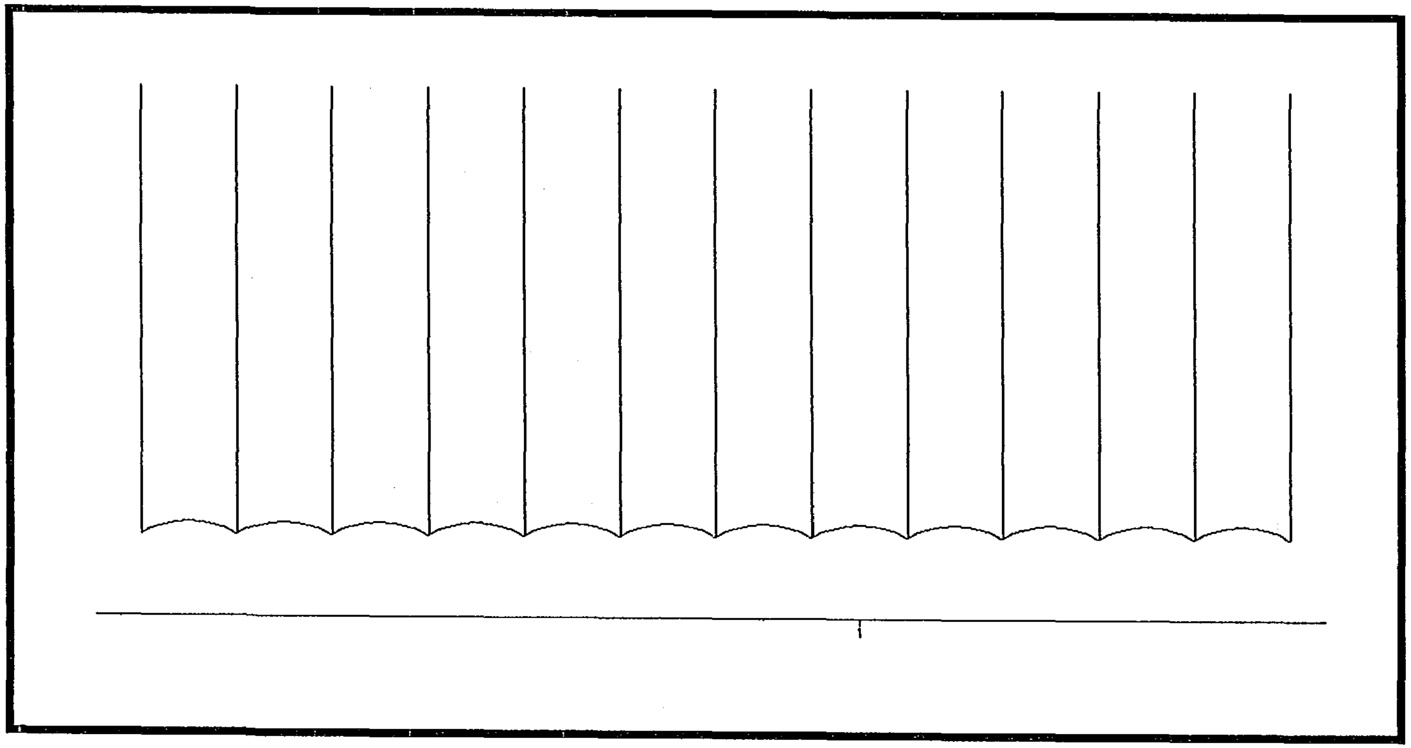


Figure 4.4: $\mathcal{R}_1$ looks like

Similarly, we can show that $\mathcal{R}_m$ for $m = 3, 4, 5,$ and 6 shares a side with $\mathcal{R}_1$ only. Next we want to show that $\mathcal{R}_n$ for $n = 7, 8, 9,$ and 10 shares a side with $\mathcal{R}_1$ only. Since $\mathcal{R}$ is connected $\mathcal{R}_n$ must share a side with $\mathcal{R}_t$ for some $t \in \{1, 2, 3, 4, ..., 10\}$ and $t \neq n$. That means there exist $0 \leq \alpha_t < \lambda_t$ and $0 \leq \alpha_n < \lambda_n$ such that $A_n S^{\alpha_n}(\mathcal{R}_0)$ shares aside with $A_t S^{\alpha_t}(\mathcal{R}_0)$. By Lemma 4.2 $A_n S^{\alpha_n} = A_t S^{\alpha_t} T$ and hence

$$A_n = A_t S^{\alpha_t} T S^{-\alpha_n} = \begin{pmatrix} b_t + \alpha_t a_t & * \\ d_t + \alpha_t c_t & * \end{pmatrix}.$$

Note that $25 | b_t + \alpha_t a_t$, because $(a_n, 125) = 25$ and $n \geq 7$. If $t \neq 1$, then $5 | a_t$ and hence $5 | b_t$. This is a contradiction. Therefore $t = 1$. Hence all of the polygons $\mathcal{R}_k$, k=2,...,10, share a side with $\mathcal{R}_1$ only. Therefore for each $k = 2, 3, \cdots, 10$, there exist an integer m_k, $0 \leq m_k < p^3$, and α_k, $0 \leq \alpha_k < \lambda_k - 1$, such that $A_k S^{\alpha_k}(\mathcal{R}_0)$ shares aside with $A_1 S^{m_k}(\mathcal{R}_0)$. By Lemma 4.2 $A_k S^{\alpha_k} = A_1 S^{m_k} T$ and

$$A_k = A_1 S^{m_k} T S^{-\alpha_k} = S^{m_k - l} T S^{-\alpha_k} = \begin{pmatrix} m_k - l & * \\ 1 & * \end{pmatrix}.$$

Also note that $m_k \neq m_n$ if $k \neq n$. Since $5 | a_k$ for each $k \in \{2, 3, ..., 10\}$, $5 | m_k - l$. If $k_1 \neq k_2$ and $k_1, k_2 \in \{2, 3, ..., 10\}$, then $p | m_{k_1} - m_{k_2}$. That means any two cusps $A_k(\infty) = m_k - l$ and $A_n(\infty) = m_n - l$ are at least 5 units apart. Hence there exists $t \in \{7, 8, 9, 10\}$ such that $5 - l < q_t = A_t(\infty) = m_t - l < 120 - l$. Now let us look $\mathcal{R}$ near the parabolic point q_t. Depending on $h := \alpha_t$, $\mathcal{R}$ looks like one of the regions in Figure 4.5.

In any of the above 5 cases the interior angle at the vertex v or w equals $4\left(\frac{\pi}{3}\right) > \pi$. This is a contradiction to the h-convexity of $\mathcal{R}$. Therefore $\Gamma^0(125)$ has no h-convex cuspidal standard fundamental domain.

Remark 4.2 *The above proposition is true for $\Gamma(p^3)$, where p is any prime $p \geq 5$. In the proof of the above proposition we used the hypothesis $p = 5$ in order to get more than 2 inequivalent cusps of width greater than 1 and this holds for any prime p greater than 5.*

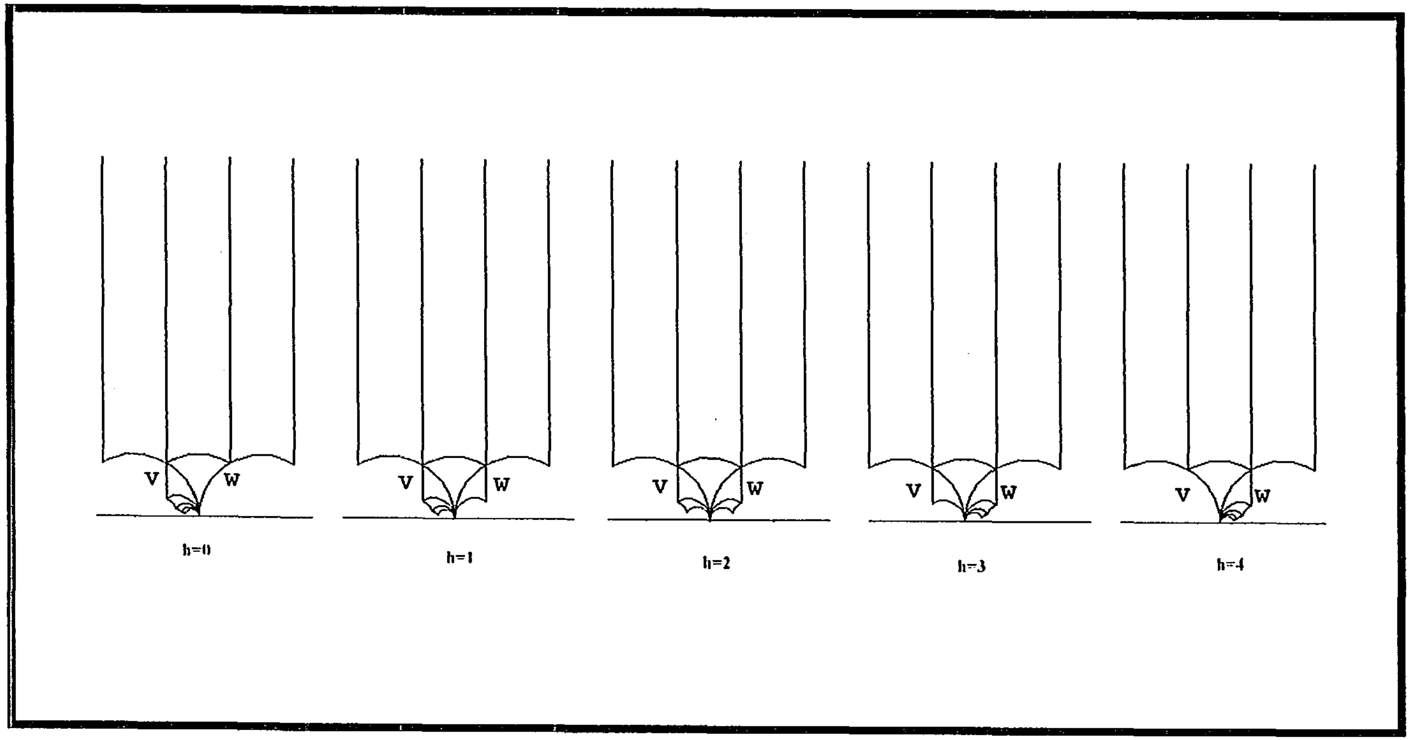


Figure 4.5: part of $\mathcal{R}$ near the cusp q_t

CHAPTER 5

MAIN RESULTS

5.1 H-convex Standard Fundamental Domains For Normal Subgroups Of The Modular Group

We know that the sets, shown in Figure 5.1, $\mathcal{R}_0 = \{\tau \in \mathbb{H} : |\tau| > 1 \ \& \ |Re(\tau)| < \frac{1}{2}\}$ and $\mathcal{R}^0 = \{\tau \in \mathbb{H} : |\tau + 1| > 1 \ \& \ 0 < Re(\tau) < \frac{1}{2}\}$ are fundamental domains for $\Gamma(1)$. Let $\mathcal{L}_1 = \{\tau \in \overline{\mathcal{R}_0} : Re(\tau) = \frac{-1}{2}\}$, $\mathcal{L}_2 = \{\tau \in \overline{\mathcal{R}_0} : Re(\tau) = \frac{1}{2}\}$,

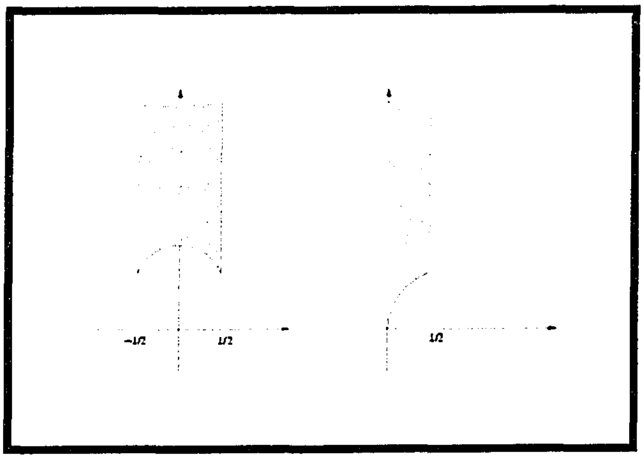


Figure 5.1: $\mathcal{R}_0$ and $\mathcal{R}^0$

and $\mathcal{C} = \{\tau \in \overline{\mathcal{R}_0} : |\tau| = 1\}$. Then $\mathcal{R}_0$ is bounded by the hyperbolic line segments $\mathcal{L}_1$, $\mathcal{C}$ and $\mathcal{L}_2$.

We also know that if Γ is a subgroup of $\Gamma(1)$ such that $\Gamma(1) = \Gamma \cdot \Sigma$, then the set

$$\mathcal{R}_\Sigma := \bigcup_{A \in \Sigma} A(\mathcal{R}_0) \tag{5.1}$$

is a fundamental domain for Γ. It is not known whether the set $\left(\overline{\mathcal{R}_\Sigma}\right)^o$ is h-convex or not. Even though the set $\left(\overline{\mathcal{R}_\Sigma}\right)^o$ may not be connected it is possible to choose Σ appropriately, as shown in Theorem 4.1, to make $\left(\overline{\mathcal{R}_\Sigma}\right)^o$ connected. Until now it has not been known whether there exists a right coset Σ of Γ in $\Gamma(1)$ such that $\left(\overline{\mathcal{R}_\Sigma}\right)^o$ is h-convex. In this chapter we will show that if Γ is a normal subgroup of $\Gamma(1)$ such that $[\Gamma(1) : \Gamma] = \mu < \infty$, then we can construct or choose a right coset decomposition Σ of Γ in $\Gamma(1)$ suitably so that the set $\left\{\overline{\mathcal{R}_\Sigma}\right\}^o$ is h-convex.

Lemma 5.1 *Suppose Γ is a normal subgroup of $\Gamma(1)$ such that $[\Gamma(1) : \Gamma] = \mu < \infty$ and either $T \in \Gamma$ or $S^2 \in \Gamma$. Then there exists a set of right coset representatives $\Sigma = \{A_1, A_2, ..., A_\mu\}$ of Γ in $\Gamma(1)$ such that $\Gamma(1) = \Gamma \cdot \Sigma$ and if $\mathcal{R} = \bigcup_{j=1}^{\mu} A_j(\mathcal{R}_0)$, then $\left(\overline{\mathcal{R}}\right)^o$ is h-convex.*

Proof: If $T \in \Gamma$ and $S^2 \in \Gamma$, observe that

$$TST = S^{-1}TS^{-1} = \underbrace{S^{-1}TS}_{\in\Gamma} \cdot \underbrace{S^{-2}}_{\in\Gamma}.$$

Hence $TST \in \Gamma$ and by normality $S \in \Gamma$. Therefore $\Gamma = \Gamma(1)$ and $\mathcal{R}_0$ is an h-convex standard fundamental domain for Γ.

Let $T \in \Gamma$ and $S^2 \notin \Gamma$. In this case we want to show that $\Gamma(1) = \Gamma \cdot \{I, S, S^2\}$ and $[\Gamma(1) : \Gamma] = 3$. $TS \notin \Gamma$, because $S^2 \notin \Gamma$. But $S^{-1}TSS^{-3} = (TS)T(TS)^{-1}T \in \Gamma$ and hence $S^{-3} \in \Gamma$. Therefore $\langle T, S^3 \rangle \subseteq \Gamma$. One can easily show that

(a) $TST = \underbrace{S^{-1}TS \cdot S^{-3}}_{\in\Gamma} \cdot S$ and $TS^{-1}T = \underbrace{STS^{-1}}_{\in\Gamma} \cdot S^2$;

(b) $TS^2T = \underbrace{TS^3T \cdot STS^{-1}}_{\in\Gamma} \cdot S^2$ and $TS^{-2}T = \underbrace{TS^{-3}T \cdot S^{-1}TS \cdot S^{-3}}_{\in\Gamma} \cdot S$;

Using the above observations and the fact that any element of $\Gamma(1)$ can be written as a word in S and T we can show that $\Gamma(1) = \Gamma \cdot \{I, S, S^2\}$. Since I, S, S^2 are inequivalent to each other modulo Γ, then $[\Gamma(1) : \Gamma] = 3$. Therefore, if we take $\Sigma = \{I,\ S,\ S^2\}$, we get an h-convex standard fundamental domain for Γ.

Let $T \notin \Gamma$ and $S^2 \in \Gamma$. We consider several cases that appear under this category. If $TS \in \Gamma$, then observe that $ST = (TSS^{-2})^{-1} \in \Gamma$ and $\langle TS, ST\rangle \subseteq \Gamma$. We want to show that $\Gamma(1) = \Gamma \cdot \{I, T\}$ and $[\Gamma(1) : \Gamma] = 2$. One can easily show that

(a) $TS^2 = \underbrace{TS \cdot ST}_{\in\Gamma} \cdot T$;

(b) $TS^3 = \underbrace{TS \cdot S^2}_{\in\Gamma} \cdot I$.

Using the above observations and the fact that any element of $\Gamma(1)$ can be written as a word in S and T we can show that $\Gamma(1) = \Gamma \cdot \{I, T\}$. Since I and T are inequivalent modulo Γ, it follows that $[\Gamma(1) : \Gamma] = 2$. Therefore taking $\Sigma = \{I,\ T\}$, we get an h-convex standard fundamental domain for Γ.

If $TS \notin \Gamma$, then we can easily see that $\Gamma(2) = \langle S^2, (ST)S^{-2}(ST)^{-1}\rangle \subseteq \Gamma$. We want to show that $\Gamma = \Gamma(2)$. First note that the mappings $I, ST, (ST)^2$ are inequivalent under the group Γ and $\Gamma(1) = \Gamma \cdot \{I, S, T, ST, STS, TST\}$. If $I \overset{\Gamma}{\sim} ST$, then $ST \in \Gamma$, which is a contradiction. If $I \overset{\Gamma}{\sim} (ST)^2$, then $(ST)^2 = TS^{-1} \in \Gamma$ and as a result $TS = TS^{-1}S^2 \in \Gamma$, which is a contradiction. If $ST \overset{\Gamma}{\sim} (ST)^2$, then $ST \in \Gamma$, which is a contradiction. Therefore $3 \leq [\Gamma(1) : \Gamma] \leq 6$. Now it remains to show that $[\Gamma(1) : \Gamma] = \mu > 3$. Since the mapping S is not equivalent to I or ST or $(ST)^2$ under the group Γ, then $\mu > 3$. Since $\mu | 6$ and $\mu > 3$, μ must be 6. Therefore $\Gamma = \Gamma(2)$ and if we take $\Sigma = \{I,\ S,\ T,\ ST,\ TST,\ STS\}$ we get an h-convex standard fundamental domain for Γ.

We have covered all the different possibilities and this completes the proof of the lemma.

q.e.d

Theorem 5.1 *Suppose that* Γ *is a normal subgroup of* $\Gamma(1)$ *such that* $[\Gamma(1) : \Gamma] = \mu < \infty$. *Then there exists a complete right coset system* $\Sigma = \{A_1, ..., A_\mu\}$ *of* Γ *in* $\Gamma(1)$ *such that*

(a) $\Gamma(1) = \Gamma \cdot \Sigma$;

(b) If $\mathcal{R} = \bigcup_{L \in \Sigma} L(\mathcal{R}_0)$, *then* $(\overline{\mathcal{R}})^o$ *is h-convex.*

Proof: Since the case $T \in \Gamma$ or $S^2 \in \Gamma$ is proved in Lemma 5.1 it is enough to prove the above theorem when $T \notin \Gamma$ and $S^2 \notin \Gamma$.

Algorithmic Construction

We define a function $\eta : \mathbb{H} \times 2^{\Gamma(1)} \to \mathbb{Z}$ as follows

$$\eta(\zeta, \Sigma) := \text{the number of elements } L \text{ of } \Sigma \text{ such that } \zeta \in \overline{L(\mathcal{R}_0)}. \tag{5.2}$$

Note that $\eta(\zeta, \Sigma)$ counts the number of modular triangles in $\mathcal{R} = \cup_{M \in \Sigma} M(\mathcal{R}_0)$ which are attached at ζ.

We want to construct a sequence of sets Σ_1, Σ_2, Σ_3, ..., Σ_m with

$$|\Sigma_1| < |\Sigma_2| < |\Sigma_3| < ... < |\Sigma_m|,$$

and $\mathcal{R}_k := \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0)$ such that $\overline{\mathcal{R}_k}$ is h-convex and each Σ_k consists of elements which are inequivalent modulo the subgroup Γ $\forall\, k \in \{1 2, 3, ..., m\}$. From now on let us fix the following convention. *The vertices of a hyperbolic polygon* $\mathbf{P} \subset \overline{\mathbb{H}}$ are the vertices, in the usual sense, of the polygon $\mathbf{P}$ that lie in $\mathbb{H}$.

STEP 1. Choose the first set Σ_1 containing only one element . Let M be an arbitrary element of $\Gamma(1)$. Let $\Sigma_1 = \{M\}$ and $\mathcal{R}_1 = \bigcup_{L \in \Sigma_1} L(\mathcal{R}_0)$. Clearly $\overline{\mathcal{R}_1}$ is h-convex.

STEP 2. The modular triangles which are adjacent to $\mathcal{R}_1$ are $MS^{-1}(\mathcal{R}_0)$, $MT(\mathcal{R}_0)$, and $MS(\mathcal{R}_0)$. Now I claim that M is inequivalent to MT under Γ. To prove this we assume the contrary. That means there exists $M_1 \in \Gamma$ such that

$$M = M_1 \cdot MT.$$

From this we can easily see that $T = M^{-1} \cdot M_1 \cdot M \in \Gamma$, because Γ is a normal subgroup. But this is a contradiction. Therefore $M \overset{\Gamma}{\nsim} MT$. If we set $\Sigma_2 = \{M,\ MT\}$, then we can see that the closure of the set $\mathcal{R}_2 = \bigcup_{L \in \Sigma_2} L(\mathcal{R}_0)$ is h-convex and $|\Sigma_2| > |\Sigma_1|$. Because of the assumption on Γ, $\mu > 2 = |\Sigma_2|$ and there exists a modular triangles adjacent to $\mathcal{R}_2$ but not equivalent to any of the modular triangles contained in $\mathcal{R}_2$. Go to the next step.

STEP k. Suppose that $\Sigma_{k-1} = \{A_1, \cdots, A_{n_{k-1}}\}$ and $\mathcal{R}_{k-1} = \cup_{j=1}^{n_{k-1}} A_j(\mathcal{R}_0)$ are obtained in STEP k-1. Then there exists $B \in \Gamma(1)$ such that $B(\mathcal{R}_0)$ is adjacent to $\mathcal{R}_{k-1}$ and $B \overset{\Gamma}{\nsim} \Sigma_{k-1}$. In this case there exists some element of Σ_{k-1}, say A_1, such that $B(\mathcal{R}_0)$ and $A_1(\mathcal{R}_0)$ share a common side. Since $B \overset{\Gamma}{\nsim} \Sigma_{k-1}$ and Γ is a normal subgroup of $\Gamma(1)$, then $I \overset{\Gamma}{\nsim} B^{-1} \cdot \Sigma_{k-1}$. Therefore, without loss of generality, we can assume B to be I, by replacing Σ_{k-1} by $B^{-1} \cdot \Sigma_{k-1}$ if necessary. Since $I(\mathcal{R}_0)$ and $A_1(\mathcal{R}_0)$ share a common side, then either $\mathcal{R}_0$ shares $\mathcal{C}$ with $A_1(\mathcal{R}_0)$ or $\mathcal{R}_0$ shares $\mathcal{L}_1$ with $A_1(\mathcal{R}_0)$ or $\mathcal{R}_0$ shares $\mathcal{L}_2$ with $A_1(\mathcal{R}_0)$. We will address each of the different possibilities one at a time.

I. $I(\mathcal{R}_0)$ shares $\mathcal{C}$ with $A_1(\mathcal{R}_0)$.

The shaded modular triangle in Figure 5.2 is contained in $\mathcal{R}_{k-1}$ and the elliptic points ρ and $\rho+1$ are vertices of $\overline{\mathcal{R}_{k-1}}$. Because of h-convexity of $\mathcal{R}_{k-1}$, $\eta(\rho, \Sigma_{k-1}) \leq 3$ and $\eta(\rho+1, \Sigma_{k-1}) \leq 3$.

Case 1. $\eta(\rho, \Sigma_{k-1}) \leq 2$ and $\eta(\rho+1, \Sigma_{k-1}) \leq 2$.

The modular triangle $\mathbf{F}_1$ in Figure 5.2 belongs to $\mathcal{R}_{k-1}$ if $\eta(\rho, \Sigma_{k-1}) = 2$. The triangle $\mathbf{F}_2$ in Figure 5.2 belongs to $\mathcal{R}_{k-1}$ if $\eta(\rho+1, \Sigma_{k-1}) = 2$. Because of

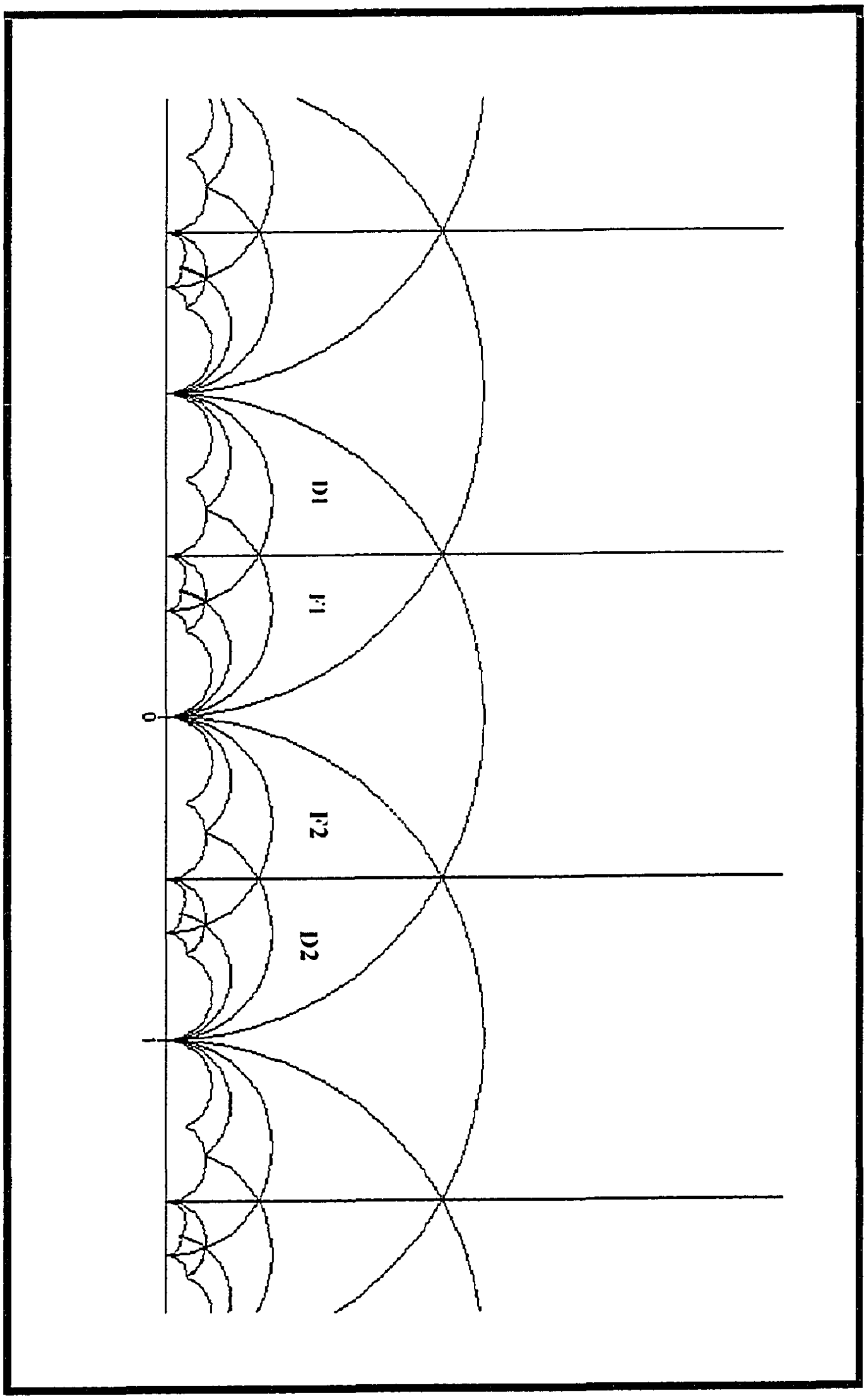


Figure 5.2:

the assumption on the values of $\eta(\rho, \Sigma_{k-1})$ and $\eta(\rho+1, \Sigma_{k-1})$, the modular triangles $\mathbf{D}_1$ and $\mathbf{D}_2$ are not contained in $\mathcal{R}_{k-1}$. Therefore

$$\mathcal{R}_{k-1} \subseteq \left\{\tau \in \mathbb{H} : |Re(\tau)| < \frac{1}{2},\ |\tau| < 1\right\}.$$

Now let

$$\Sigma_k = \Sigma_{k-1} \cup \{I\} \quad \text{and} \quad \mathcal{R}_k = \mathcal{R}_{k-1} \cup I(\mathcal{R}_0).$$

Then

(a) any two distinct elements of Σ_k are inequivalent under Γ;

(b) all the vertices of $\overline{\mathcal{R}_{k-1}}$ remains vertices of $\overline{\mathcal{R}_k}$, but the interior angles at ρ and $\rho + 1$ get modified and these are the only ones that are modified. Depending on the values of $\eta(\rho, \Sigma_{k-1})$ and $\eta(\rho+1, \Sigma_{k-1})$, the interior angle at ρ and $\rho + 1$ is either π or $\frac{2\pi}{3}$. By Theorem 3.5, $\overline{\mathcal{R}_k}$ is h-convex;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 1$.

Remark 5.1 *Since Σ_{k-1} might have been modified as mentioned previously, Σ_k may not contain Σ_{k-1}.*

GO TO STEP k+1.

Case 2. $\eta(\rho, \Sigma_{k-1}) = 3$ and $\eta(\rho+1, \Sigma_{k-1}) \leq 2$.
From the assumption $\eta(\rho, \Sigma_{k-1}) = 3$ we can conclude that $T,\ TS,\ TST \in \Sigma_{k-1}$.

The shaded modular triangles in Figure 5.3 are contained in $\mathcal{R}_{k-1}$, but the modular triangle denoted by $\mathbf{F}$ may not belong to $\mathcal{R}_{k-1}$, depending on the value of $\eta(\rho+1, \Sigma_{k-1})$. Therefore

$$\mathcal{R}_{k-1} \subseteq \left\{\tau \in \mathbb{H} : Re(\tau) < \frac{1}{2}, |\tau| < 1\right\}.$$

To get the process going we have to consider the following different cases.

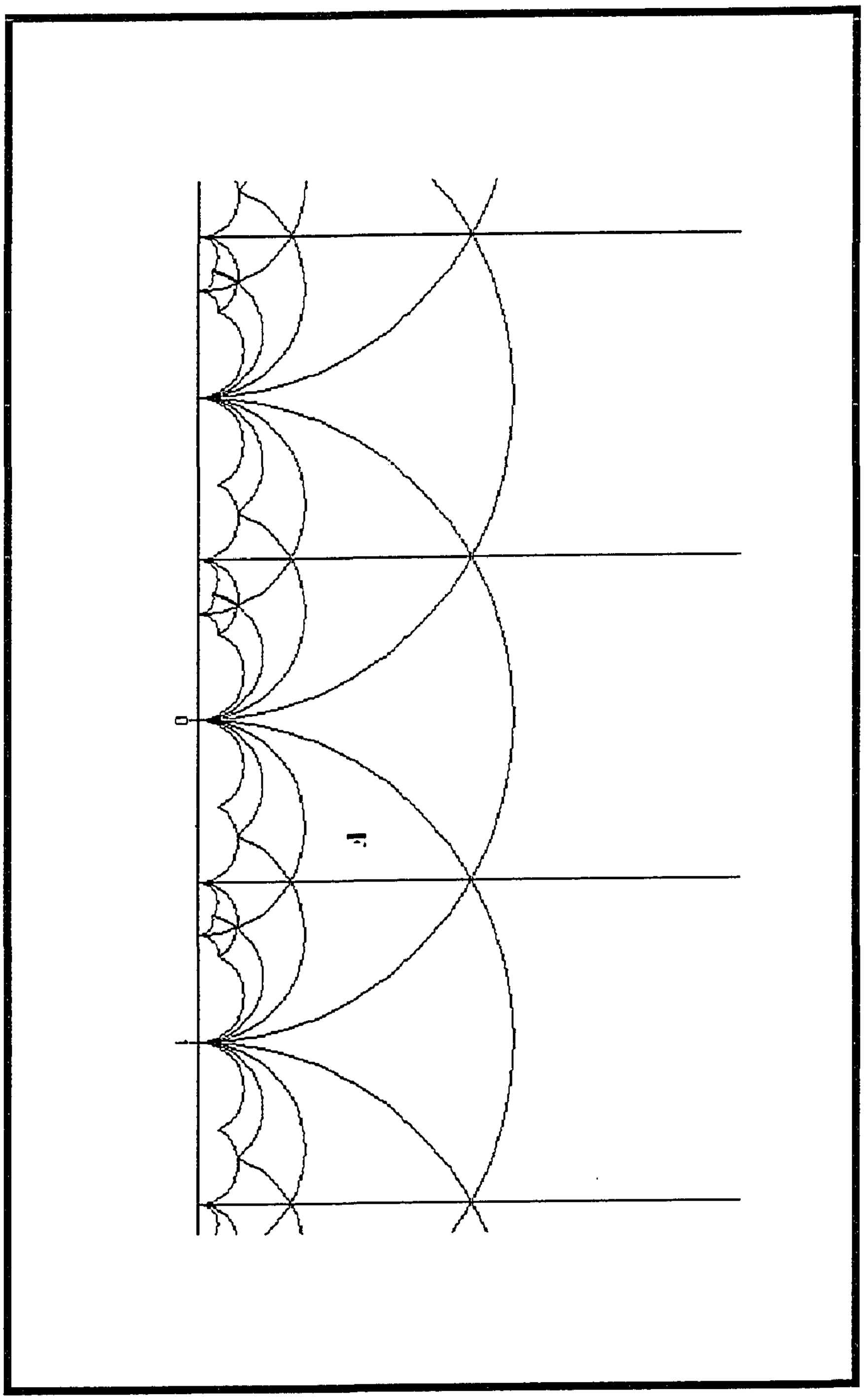


Figure 5.3:

Subcase 2.1 $S^{-1} \overset{\Gamma}{\sim} \Sigma_{k-1}$.

Then there exist $A_l \in \Sigma_{k-1}$ and $M \in \Gamma$ such that $S^{-1} = MA_l$ with $M \neq I$. Consider the set

$$\Sigma_{k-1}^* = \{MA_1, MA_2, ..., MA_{n_{k-1}}\} = M \cdot \Sigma_{k-1}$$

and $\mathcal{R}_{k-1}^* = M(\mathcal{R}_{k-1}) = \bigcup_{A \in \Sigma_{k-1}^*} A(\mathcal{R}_0)$. Since $\mathcal{R}_{k-1}$ is a subset of a fundamental domain for Γ and $M \in \Gamma$, then $\mathcal{R}_{k-1} \cap \mathcal{R}_{k-1}^* = \emptyset$. Also,

$$\mathcal{R}_{k-1}^* \cap I(\mathcal{R}_0) = \emptyset, \text{ and } S^{-1}(\mathcal{R}_0) \subseteq \mathcal{R}_{k-1}^*.$$

The shaded modular triangle in Figure 5.4 belongs to $\mathcal{R}_{k-1}^*$ and the modular triangles denoted by **D** are not contained in $\mathcal{R}_{k-1}^*$. Therefore, by h-convexity of $\overline{\mathcal{R}_{k-1}^*}$,

$$\mathcal{R}_{k-1}^* \subseteq \left\{\tau \in \mathbb{H} : Re(\tau) < \frac{-1}{2}, |\tau| > 1\right\}.$$

Now let $\Sigma_k = \Sigma_{k-1}^* \cup \{I\}$ and $\mathcal{R}_k = \mathcal{R}_{k-1}^* \cup I(\mathcal{R}_0)$. Then

(a) any two distinct elements of Σ_k are not equivalent modulo Γ;

(b) all the vertices of $\overline{\mathcal{R}_{k-1}^*}$ remains vertices of $\overline{\mathcal{R}_k}$ and the new polygon has one more vertex, namely $\rho + 1$. Moreover, the interior angle at ρ gets modified and depending on the value of $\eta(\rho, \Sigma_{k-1}^*)$, the interior angle at ρ is either π or $\frac{2\pi}{3}$. Also the interior angle at $\rho + 1$ is $\frac{\pi}{3}$. By Theorem 3.5, $\overline{\mathcal{R}_k}$ is h-convex;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 1$.

GO TO STEP k+1.

Subcase 2.2 $S^{-1} \overset{\Gamma}{\nsim} \Sigma_{k-1}$ and $S^{-1}T \overset{\Gamma}{\sim} \Sigma_{k-1}$.

Then there exist $A_l \in \Sigma_{k-1}$ and $M \in \Gamma$ such that $S^{-1}T = MA_l$ with $M \neq I$ and $A_l \notin \{T, TS, TST\}$, because if $A_l \in \{T, TS, TST\}$, then either T or S^2 will be in the group Γ, and this is a contradiction. Let

$$\Sigma_{k-1}^* = \{MA_1, MA_2, ..., MA_{n_{k-1}}\} = M \cdot \Sigma_{k-1} \text{ and } \mathcal{R}_{k-1}^* = M(\mathcal{R}_{k-1}).$$

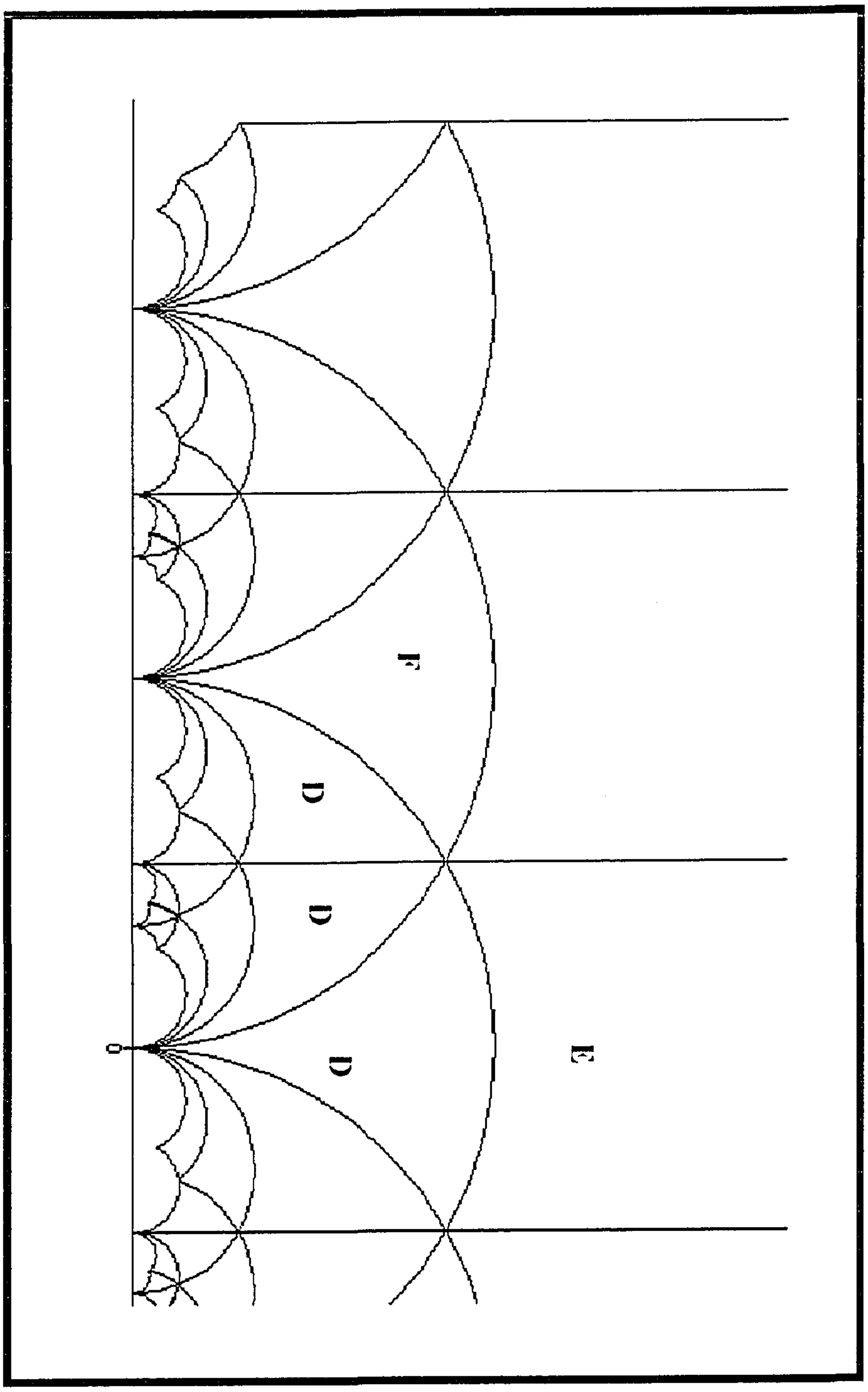


Figure 5.4:

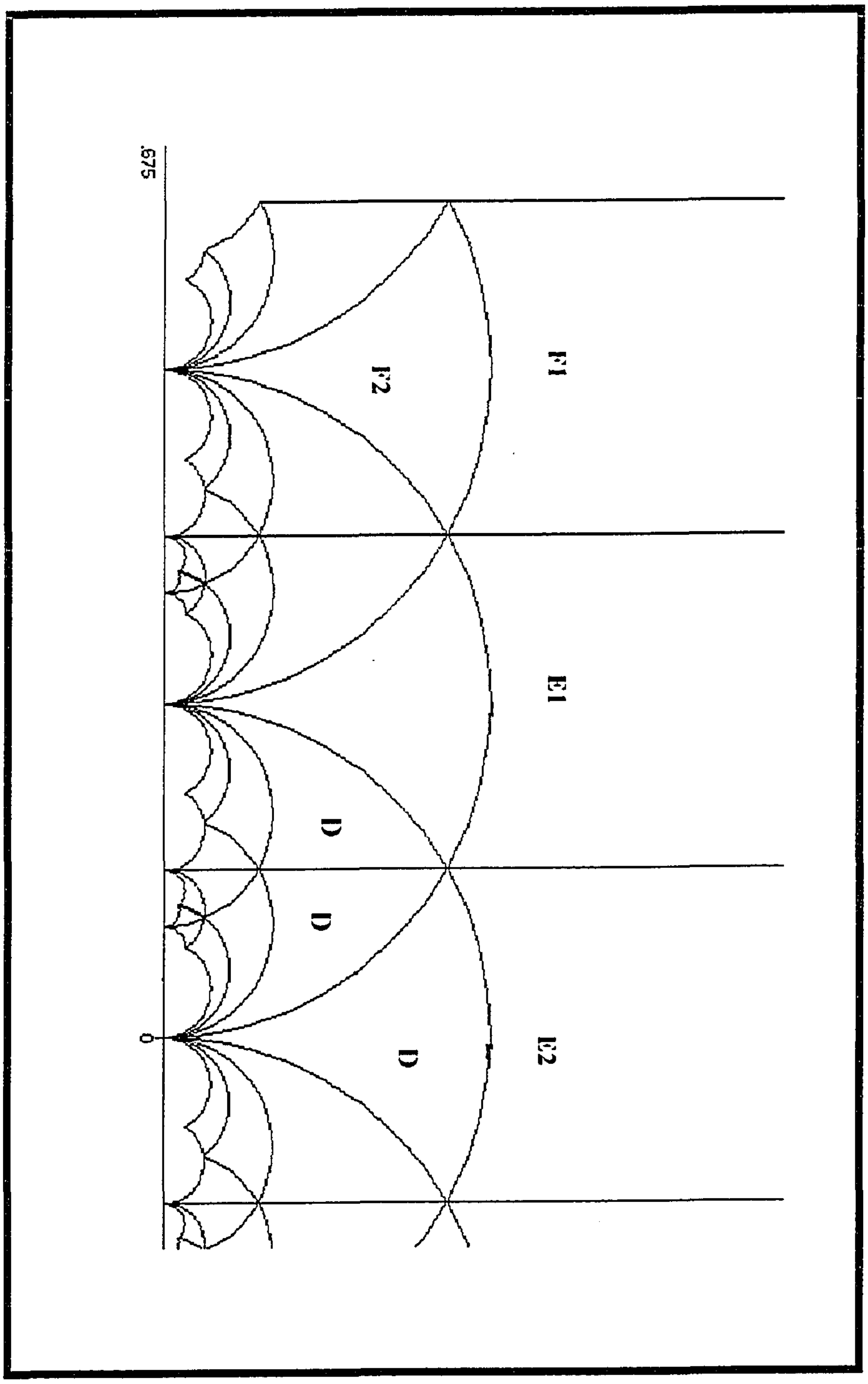


Figure 5.5:

Note that $I \overset{\Gamma}{\not\sim} \Sigma^*_{k-1}$, $S^{-1} \overset{\Gamma}{\not\sim} \Sigma^*_{k-1}$ and $I \overset{\Gamma}{\not\sim} S^{-1}$.

The shaded modular triangle in Figure 5.5 is contained in $\mathcal{R}^*_{k-1}$. Since $\mathcal{R}_{k-1} \cap \mathcal{R}^*_{k-1} = \emptyset$, the modular triangles denoted by **D** are not contained in $\mathcal{R}^*_{k-1}$.

Since I and S^{-1} are inequivalent to Σ_{k-1} and also to Σ^*_{k-1}, the modular triangles $\mathbf{E}_1$ and $\mathbf{E}_2$ are not contained in $\mathcal{R}^*_{k-1}$.

Since $\overline{\mathcal{R}^*_{k-1}}$ is h-convex and $\rho - 1$ is not an interior point, the modular triangles $\mathbf{F}_1$ and $\mathbf{F}_2$ are not contained in $\mathcal{R}^*_{k-1}$.

Therefore

$$S^{-1}T(\mathcal{R}_0) \subseteq \mathcal{R}^*_{k-1} \subseteq \{\tau \in \mathbb{H} : |\tau + 1| < 1 \ \&\ |\tau| > 1\}.$$

If $\Sigma^{**}_{k-1} = \Sigma^*_{k-1} \setminus \{S^{-1}T\}$, then the closure of the set $\mathcal{R}^{**}_{k-1} = \bigcup_{L \in \Sigma^{**}_{k-1}} L(\mathcal{R}_0)$ is h-convex. If we remove a triangle which has at least one vertex not shared by any other triangle from a convex polygon, then the resulting polygon will also be convex. Therefore $M^{-1}(\mathcal{R}^{**}_{k-1}) = \mathcal{R}_{k-1} \setminus A_l(\mathcal{R}_0)$ has h-convex closure. Also note that the set of vertices of $\overline{\mathcal{R}^{**}_{k-1}}$ is the same as the set of vertices of $\overline{\mathcal{R}^*_{k-1}}$ minus $\{\rho\}$. Now let

$$\Sigma_k = (\Sigma_{k-1} \setminus \{A_l\}) \cup \{I, S^{-1}, S^{-1}T\}$$

and

$$\mathcal{R}_k = \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0).$$

Clearly, the shaded region in Figure 5.6 is contained in $\mathcal{R}_k$. The vertex ρ of $\overline{\mathcal{R}_{k-1}}$ now becomes an interior point of $\overline{\mathcal{R}_k}$. But the vertex $M^{-1}(\rho)$ of $\overline{\mathcal{R}_{k-1}}$ is not a vertex of $\overline{\mathcal{R}_k}$. Therefore all the vertices of $\overline{\mathcal{R}_{k-1}}$ except $M^{-1}(\rho)$ together with the newly added vertex $\rho - 1$ forms the set of vertices of $\overline{\mathcal{R}_k}$. Then

(a) any two distinct elements of Σ_k are not equivalent modulo Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 2$.

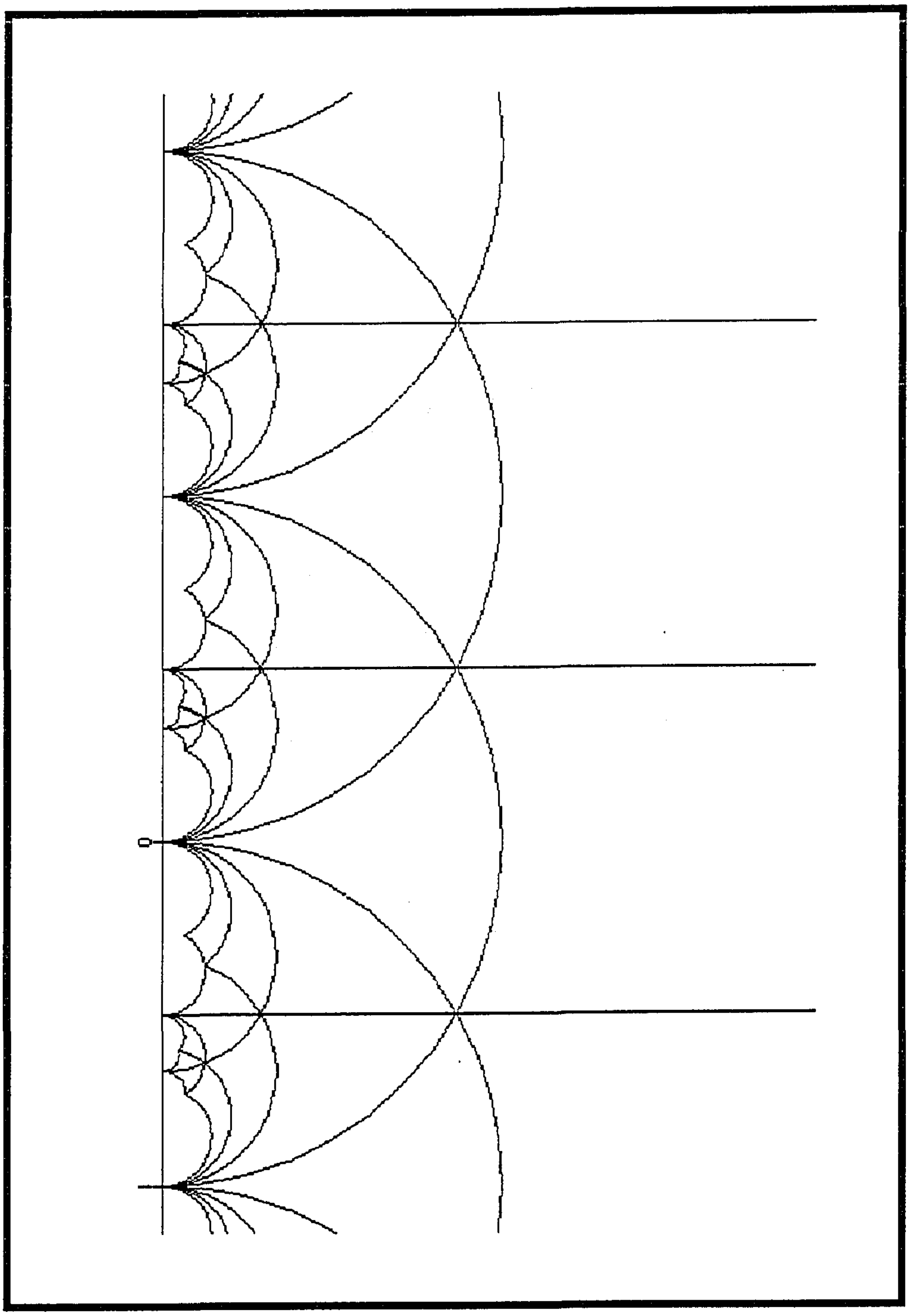


Figure 5.6:

GO TO STEP k+1.

Subcase 2.3 $S^{-1} \overset{\Gamma}{\not\sim} \Sigma_{k-1}$ and $S^{-1}T \overset{\Gamma}{\not\sim} \Sigma_{k-1}$.

In this case we have I, $S^{-1}T$, and S^{-1} are inequivalent to Σ_{k-1} under Γ. Moreover, any two distinct elements of $\{I, S^{-1}, S^{-1}T\}$ are inequivalent under Γ. Let

$$\Sigma_k := \Sigma_{k-1} \cup \{I,\ S^{-1}, S^{-1}T\}, \quad \text{and} \quad \mathcal{R}_k = \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0).$$

Note that the vertex ρ of $\overline{\mathcal{R}_{k-1}}$ now becomes an interior point of $\overline{\mathcal{R}_k}$. Therefore the vertices of $\overline{\mathcal{R}_k}$ are the vertices of $\overline{\mathcal{R}_{k-1}}$ together with $\{\rho - 1\}$ minus $\{\rho\}$. Then

(a) any two distinct elements of Σ_k are inequivalent modulo Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 3$.

GO TO STEP k+1.

Case 3. $\eta(\rho, \Sigma_{k-1}) \leq 2$ and $\eta(\rho+1, \Sigma_{k-1}) = 3$.

Repeat Case 2 replacing S^{-1} by S and $S^{-1}T$ by ST.

Case 4. $\eta(\rho, \Sigma_{k-1}) = 3$ and $\eta(\rho+1, \Sigma_{k-1}) = 3$.

Clearly $\{T, TS, TST, TS^{-1}, STS\} \subset \Sigma_{k-1}$, i.e, the shaded modular triangles in Figure 5.7 are contained in $\overline{\mathcal{R}_{k-1}}$. Because of h-convexity

$$\overline{\mathcal{R}_{k-1}} \subset \{\tau \in \mathbb{H} : |\tau| \leq 1\}.$$

Subcase 4.1 $S^{-1} \overset{\Gamma}{\sim} \Sigma_{k-1}$.

Repeat Subcase 2.1 .

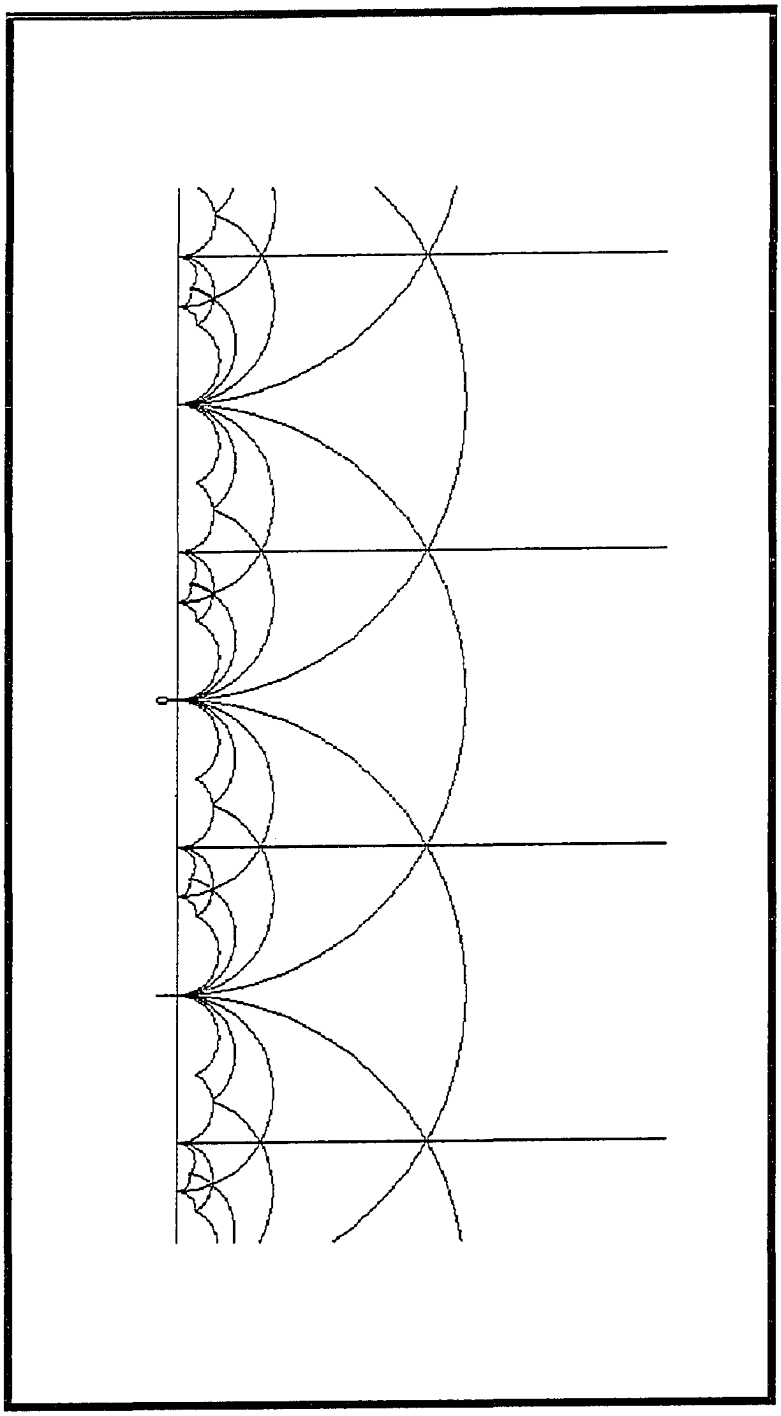


Figure 5.7:

Subcase 4.2 $S^{-1} \overset{\Gamma}{\nsim} \Sigma_{k-1}$ and $S \overset{\Gamma}{\sim} \Sigma_{k-1}$.

Repeat Subcase 3.1 .

Subcase 4.3 $S^{-1} \overset{\Gamma}{\nsim} \Sigma_{k-1}$ and $S \overset{\Gamma}{\nsim} \Sigma_{k-1}$.

In this case we have I, S^{-1}, and S are inequivalent to Σ_{k-1} under Γ.

Subcase 4.3.1 $S^{-1}T \overset{\Gamma}{\sim} \Sigma_{k-1}$.

Then there exist $A_l \in \Sigma_{k-1}$ and $M \in \Gamma$ such that $S^{-1}T = M \cdot A_l$. Let $\Sigma^*_{k-1} = M \cdot \Sigma_{k-1}$ and $\mathcal{R}^*_{k-1} = M(\mathcal{R}_{k-1})$. The shaded modular triangle in Figure 5.8 is contained in $\mathcal{R}^*_{k-1}$. By Lemma 4.2 $\mathcal{R}^*_{k-1} \cap \mathcal{R}_{k-1} = \emptyset$.

The modular triangles denoted by **D**, in Figure 5.8, are not contained in $\mathcal{R}^*_{k-1}$, because $\mathcal{R}_{k-1} \cap \mathcal{R}^*_{k-1} = \emptyset$.

The modular triangles $\mathbf{E}_1$ and $\mathbf{E}_2$, in Figure 5.8, are not contained in $\mathcal{R}^*_{k-1}$, because I and S^{-1} are inequivalent to Σ_{k-1} and also to Σ^*_{k-1}.

Therefore

$$\mathcal{R}^*_{k-1} \subset \{\tau \in \mathbb{H} : |\tau| > 1, \ \ |\tau + 1| < 1\}$$

and $\overline{\mathcal{R}^*_{k-1} \setminus S^{-1}T(\mathcal{R}_0)}$ is h-convex. If we let $\Sigma^{**}_{k-1} = \Sigma_{k-1} \setminus \{A_l\}$ and $\mathcal{R}^{**}_{k-1} = \bigcup_{L \in \Sigma^{**}_{k-1}} L(\mathcal{R}_0)$, then $\overline{\mathcal{R}^{**}_{k-1}}$ is h-convex and $S^{-1}T \overset{\Gamma}{\nsim} \Sigma^{**}_{k-1}(\bmod \Gamma)$.

If $ST \overset{\Gamma}{\sim} \Sigma^{**}_{k-1}$, then there exists $A_t \in \Sigma^{**}_{k-1}$ and $M_1 \in \Gamma$ such that $ST = M_1 \cdot A_t$. As in the previous case, if $\Sigma^{***}_{k-1} = \Sigma^{**}_{k-1} \setminus \{A_t\}$ and $\mathcal{R}^{***}_{k-1} = \bigcup_{L \in \Sigma^{***}_{k-1}} L(\mathcal{R}_0)$, then $\overline{\mathcal{R}^{***}_{k-1}}$ is h-convex. Moreover, $ST \overset{\Gamma}{\nsim} \Sigma^{***}_{k-1}$ and $|\Sigma^{***}_{k-1}| = |\Sigma_{k-1}| - 2 > 0$. Therefore

$$I, S^{-1}, S^{-1}T, S, ST \overset{\Gamma}{\nsim} \Sigma^{***}_{k-1},$$

and because of the assumption on Γ any two pair of elements of $\{I, S^{-1}, S^{-1}T, S, ST\}$ are inequivalent under Γ.

Let

$$\Sigma_k = \Sigma^{***}_{k-1} \cup \{I, S^{-1}, S^{-1}T, S, ST\}, \ \text{ and } \mathcal{R}_k = \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0).$$

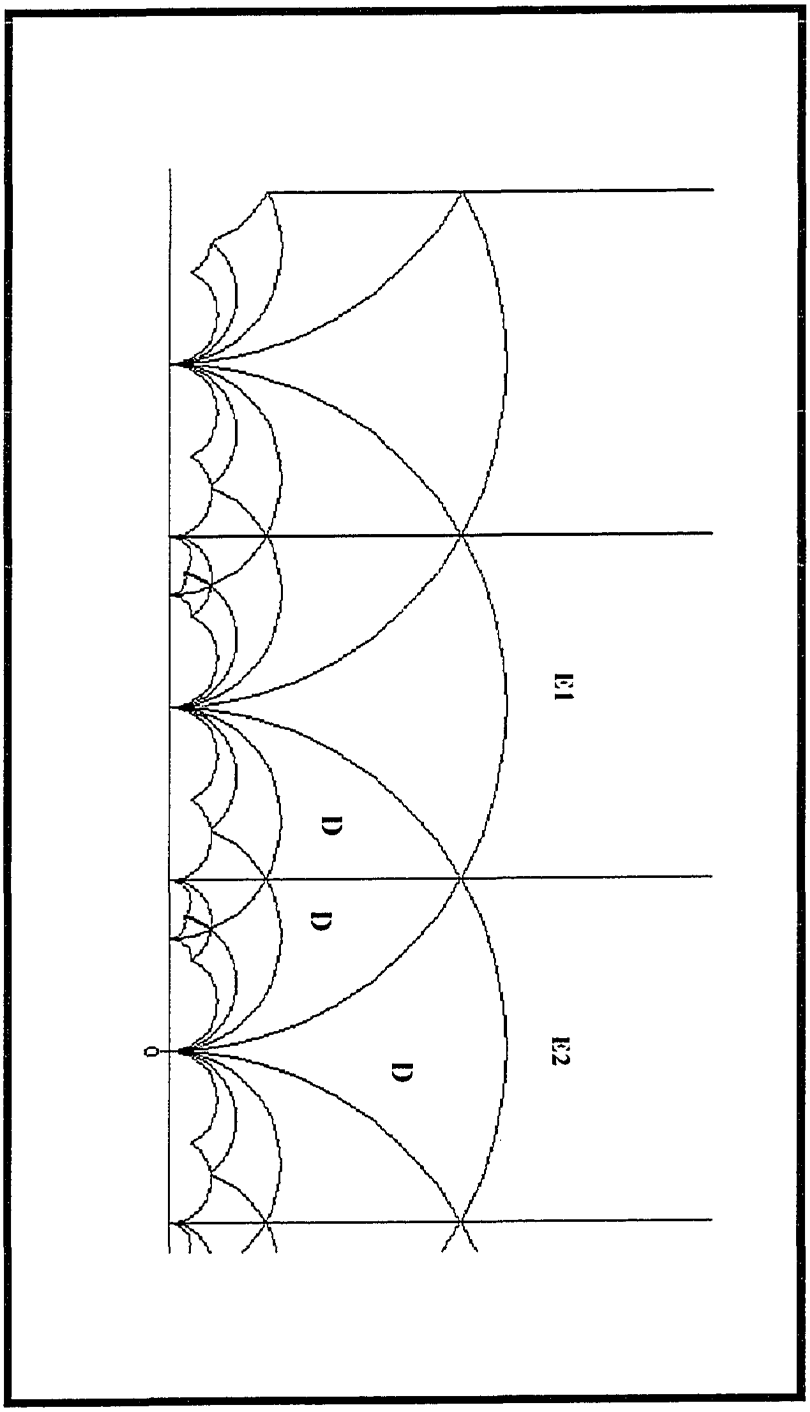


Figure 5.8:

Note that the shaded modular triangles in Figure 5.9 are contained in $\mathcal{R}_k$. Also note that

(a) any two distinct elements of Σ_k are inequivalent under Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 3$.

GO TO STEP k+1.

If $ST \overset{\Gamma}{\not\sim} \Sigma_{k-1}^{**}$, then $I, S^{-1}, S^{-1}T, S, ST \overset{\Gamma}{\not\sim} \Sigma_{k-1}^{**}$, and because of the assumption on Γ any two pair of elements of $\{I, S^{-1}, S^{-1}T, S, ST\}$ are inequivalent under Γ. Let

$$\Sigma_k = \Sigma_{k-1}^{**} \cup \{I, S^{-1}, S^{-1}T, S, ST\} \text{ and } \mathcal{R}_k = \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0).$$

Then

(a) any two distinct elements of Σ_k are inequivalent under Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 4$.

GO TO STEP k+1.

Subcase 4.3.2 $\quad S^{-1}T \overset{\Gamma}{\not\sim} \Sigma_{k-1}$.

If $ST \overset{\Gamma}{\sim} \Sigma_{k-1}$, then there exists $A_t \in \Sigma_{k-1}$ and $M_1 \in \Gamma$ such that $ST = M_1 \cdot A_t$. As in the previous case, if $\Sigma_{k-1}^{*} = \Sigma_{k-1} \setminus \{A_t\}$ and $\mathcal{R}_{k-1}^{*} = \bigcup_{L \in \Sigma_{k-1}^{*}} L(\mathcal{R}_0)$, then $\overline{\mathcal{R}_{k-1}^{*}}$ is h-convex. Moreover, $ST \overset{\Gamma}{\not\sim} \Sigma_{k-1}^{*}$ and $|\Sigma_{k-1}^{*}| = |\Sigma_{k-1}| - 1 > 0$. Hence we have $I, S^{-1}, S^{-1}T, S, ST \overset{\Gamma}{\not\sim} \Sigma_{k-1}^{*}$ and because of the assumption on Γ, any two pair of elements of $\{I, S^{-1}, S^{-1}T, S, ST\}$ are inequivalent modulo Γ. Let

$$\Sigma_k = \Sigma_{k-1}^{*} \cup \{I, S^{-1}, S^{-1}T, S, ST\} \text{ and } \mathcal{R}_k = \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0).$$

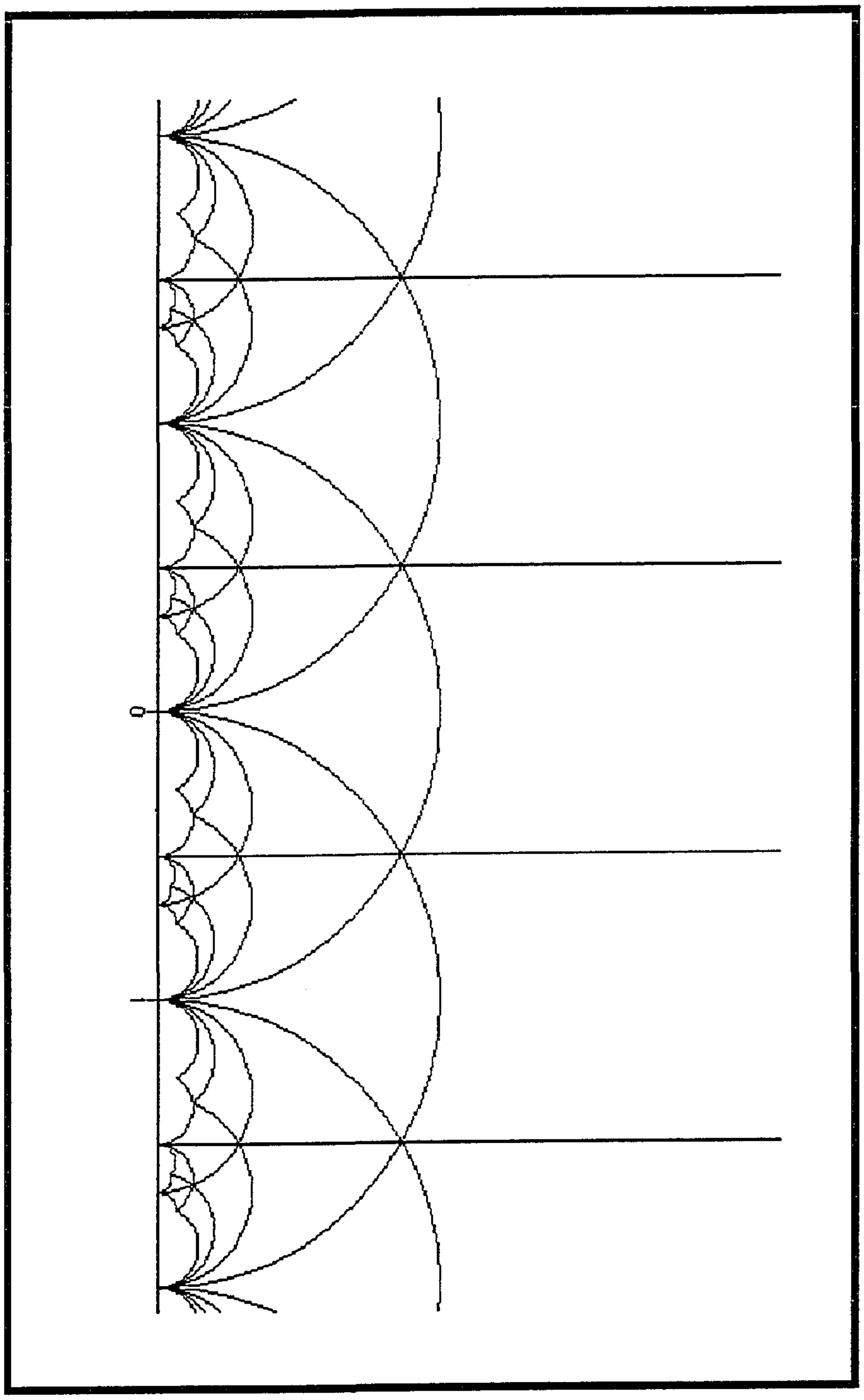


Figure 5.9:

Then

(a) any two distinct elements of Σ_k are inequivalent under Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 4$.

GO TO STEP k+1.

If $ST \overset{\Gamma}{\not\sim} \Sigma_{k-1}$, then I, S^{-1}, $S^{-1}T$, S, ST are inequivalent to Σ_{k-1} under Γ and because of the assumption on Γ, any two pair of elements of $\{I, S^{-1}, S^{-1}T, S, ST\}$ are inequivalent under Γ. Let

$$\Sigma_k = \Sigma_{k-1} \cup \{I, S^{-1}, S^{-1}T, S, ST\} \text{ and } \mathcal{R}_k = \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0).$$

Then

(a) any two distinct elements of Σ_k are inequivalent under Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = |\Sigma_{k-1}| + 5$.

GO TO STEP k+1.

II. $I(\mathcal{R}_0)$ shares $\mathcal{L}_1$ with $A_1(\mathcal{R}_0)$.

The shaded modular triangle in Figure 5.10 is contained in $\overline{\mathcal{R}_{k-1}}$, but the modular triangles denoted by **D** are not contained in $\mathcal{R}_{k-1}$. Therefore, by h-convexity,

$$\mathcal{R}_{k-1} \subset \left\{\tau \in \mathbb{H} : Re(\tau) < \frac{-1}{2}\right\}.$$

Since the elliptic point ρ is a vertex of the hyperbolic polygon $\overline{\mathcal{R}_{k-1}}$, it follows that $\eta(\rho, \Sigma_{k-1}) \leq 3$, by Theorem 3.5. There are two cases we have to consider.

Case 1. $\eta(\rho, \Sigma_{k-1}) \leq 2$.

Let $\Sigma_k = \Sigma_{k-1} \cup \{I\}$ and $\mathcal{R}_k = \mathcal{R}_{k-1} \cup I(\mathcal{R}_0)$. Then

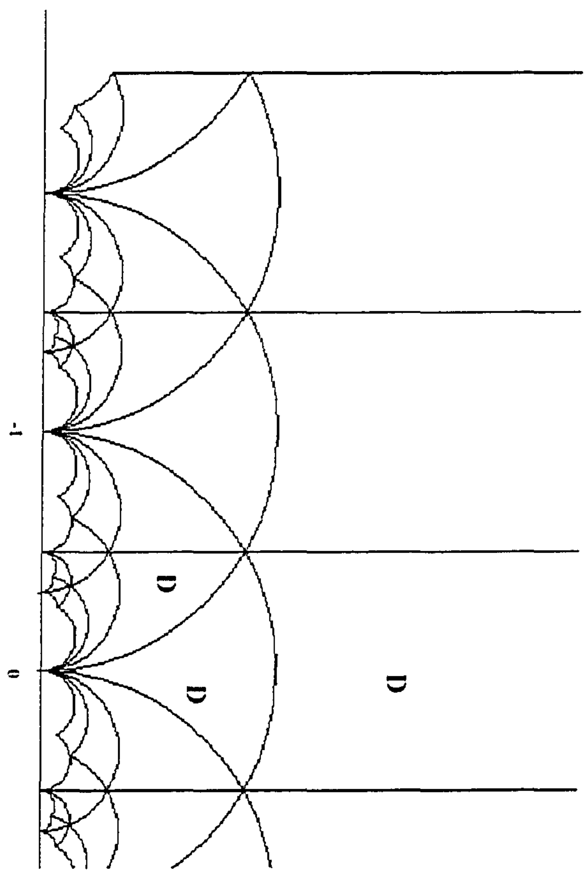


Figure 5.10:

(a) any two distinct elements of Σ_k are inequivalent under Γ;

(b) $\overline{\mathcal{R}_k}$ is h-convex, by Theorem 3.5;

(c) $|\Sigma_k| = n_{k-1} + 1$.

GO TO STEP k+1.

Case 2. $\eta(\rho, \Sigma_{k-1}) = 3$.

By our assumption $S^{-1}, S^{-1}T, TST \in \Sigma_{k-1}$ and hence the shaded modular triangles in Figure 5.11 are contained in $\mathcal{R}_{k-1}$.

Moreover, the modular triangle $TS(\mathcal{R}_0)$ shares a side $TS(\mathcal{C})$ with $\mathcal{R}_{k-1}$.

Subcase 2.1 $TS \stackrel{\Gamma}{\nsim} \Sigma_{k-1}$.

In this case apply (I) with TS in the place of I.

Subcase 2.2 $TS \stackrel{\Gamma}{\sim} \Sigma_{k-1}$.

There exist $M \in \Gamma$ and $A_l \in \Sigma_{k-1}$ such that $TS = M \cdot A_l$. Let

$$\Sigma_{k-1}^{*} = M \cdot \Sigma_{k-1}, \text{ and } \mathcal{R}_{k-1}^{*} = M(\mathcal{R}_{k-1}).$$

Since $I \stackrel{\Gamma}{\nsim} \Sigma_{k-1}$, then $I \stackrel{\Gamma}{\nsim} \Sigma_{k-1}^{*}$. There are two cases we have to consider.

(i) . $T \stackrel{\Gamma}{\sim} \Sigma_{k-1}^{*}$.

In this case there exist $M_1 \in \Gamma$ and $A_t \in \Sigma_{k-1}^{*}$ such that $T = M_1 \cdot A_t = (M_1 M) \cdot A_t$, and as a result we get $T \stackrel{\Gamma}{\sim} \Sigma_{k-1}$. Let

$$\Sigma_{k-1}^{**} = M_1 \cdot \Sigma_{k-1}^{*}, \text{ and } \mathcal{R}_{k-1}^{**} = M_1(\mathcal{R}_{k-1}^{*}).$$

Then $I \stackrel{\Gamma}{\nsim} \Sigma_{k-1}^{**}$ and $I(\mathcal{R}_0)$ shares a side $\mathcal{C}$ with $\mathcal{R}_{k-1}^{**}$. Now we can apply the first case (I) with Σ_{k-1}^{**} in the place of Σ_{k-1}.

(ii) . $T \stackrel{\Gamma}{\nsim} \Sigma_{k-1}^{*}$.

The shaded modular triangle in Figure 5.12 is contained in $\mathcal{R}_{k-1}^{*}$. The modular triangles $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are not contained in $\mathcal{R}_{k-1}^{*}$, because

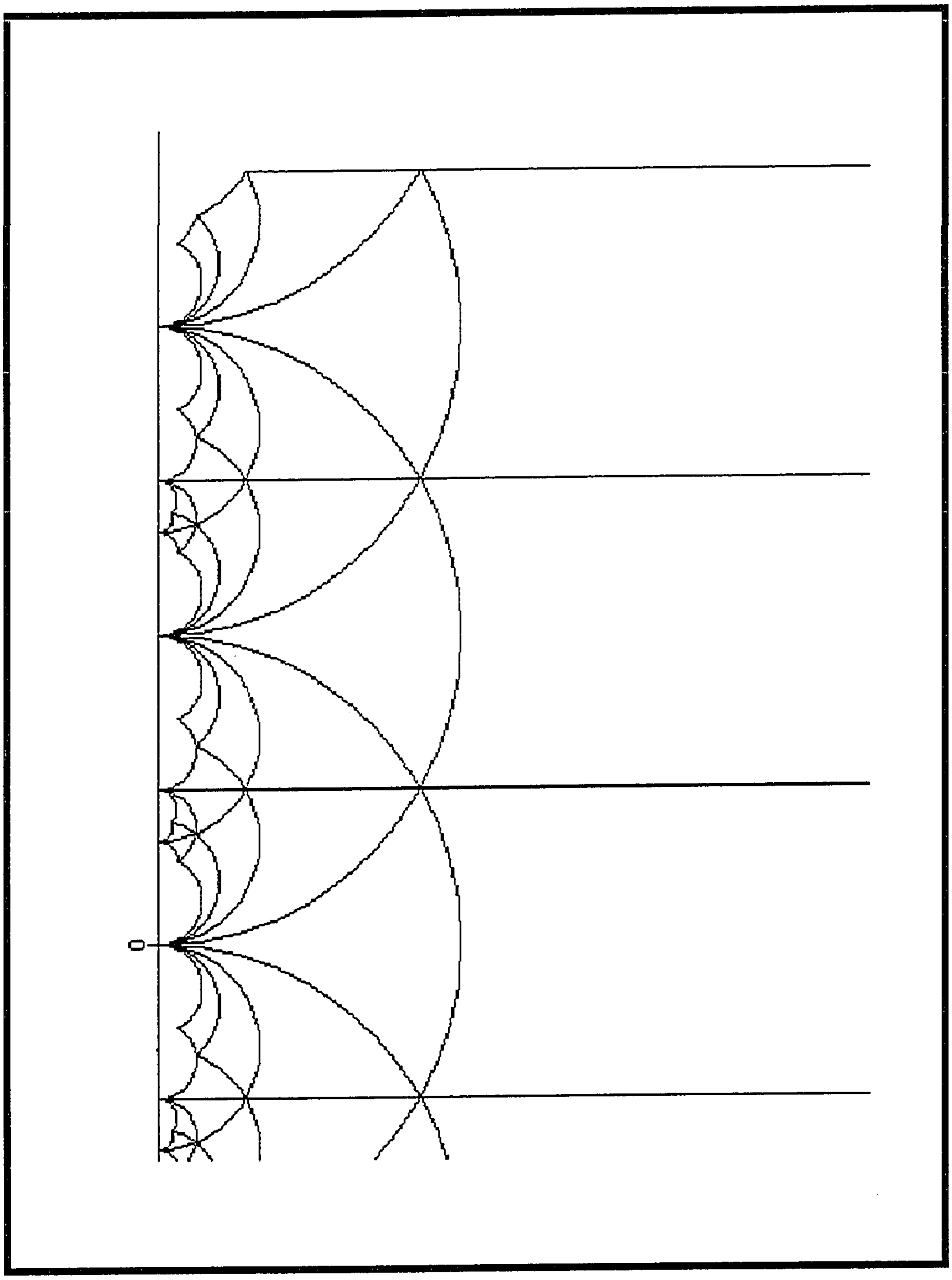


Figure 5.11:

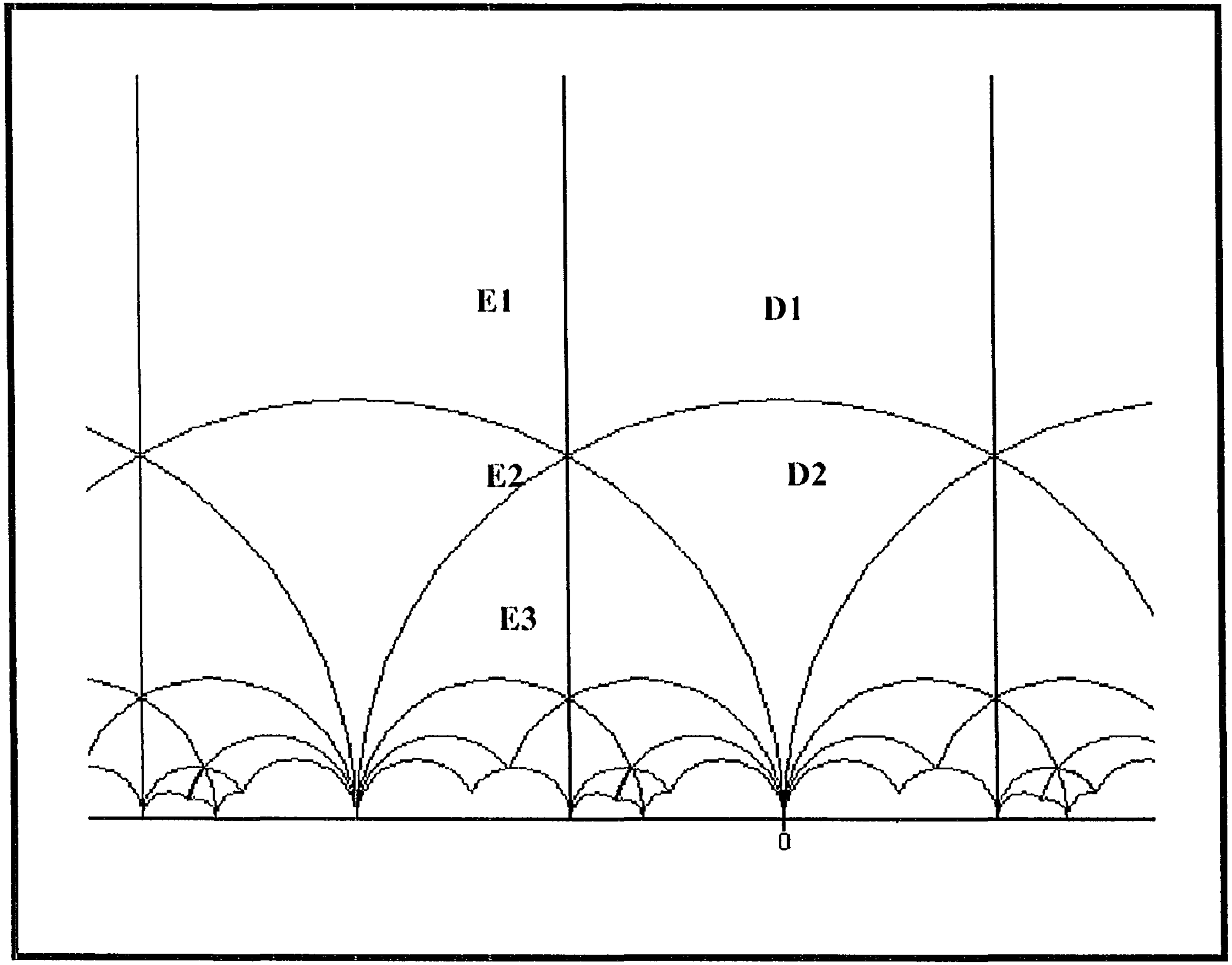


Figure 5.12:

$\mathcal{R}_{k-1} \cap \mathcal{R}^*_{k-1} = \emptyset$. Since I and T are inequivalent to Σ^*_{k-1}, the modular triangles $\mathbf{D}_1$ and $\mathbf{D}_2$ are not contained in $\mathcal{R}^*_{k-1}$. Therefore

$$\mathcal{R}^*_{k-1} \subset \left\{ \tau \in \mathbb{H} : Re(\tau) > \frac{-1}{2} \;\&\; |\tau + 1| < 1 \right\}.$$

Let $\Sigma_k := \Sigma^*_{k-1} \cup \{T\}$ and $\mathcal{R}_k := \bigcup_{L \in \Sigma_k} L(\mathcal{R}_0)$. We can easily show that

(a) any two distinct elements of Σ_k are inequivalent modulo Γ;

(b) $\mathcal{R}_k$ has hconvex closure interior;

(c) $|\Sigma_k| = n_{k-1} + 1$.

GO TO STEP k+1.

III. $I(\mathcal{R}_0)$ **shares** $\mathcal{L}_2$ **with** $A_1(\mathcal{R}_0)$.
This case is similar to (II).

Since the index $[\Gamma(1) : \Gamma] = \mu < \infty$ the process has to terminate after a finite number of steps, say after the m^{th} step. The process terminates if either $\Sigma_m = [\Gamma(1) : \Gamma]$ or every modular triangle which is adjacent to $\mathcal{R}_m$ is equivalent to some modular triangle in $\mathcal{R}_m$. By Corollary 4.1 these two statements are equivalent.

Q.E.D

Remark 5.2 *Theorem 5.1 remains true if we replace the modular group* $\Gamma(1)$ *by the discrete Hecke group* $H(\lambda)$*, where* $\lambda = 2cos(\frac{\pi}{q})$*. But the proof is messier than the one given above, because at each elliptic point of order* q *there are* $2q$ *replicas of* $\mathcal{R}_\lambda$ *attached.*

Example 5.1 *Consider the normal subgroup* $\Gamma(4)$ *of* $\Gamma(1)$*. We know that*

(i) $[\Gamma(1) : \Gamma(4)] = 24$*;*

(ii) The parabolic class number of $\Gamma(4)$ *is 6.*

An H-convex SFD for $\Gamma(4)$ *is given in Figure 5.13.*

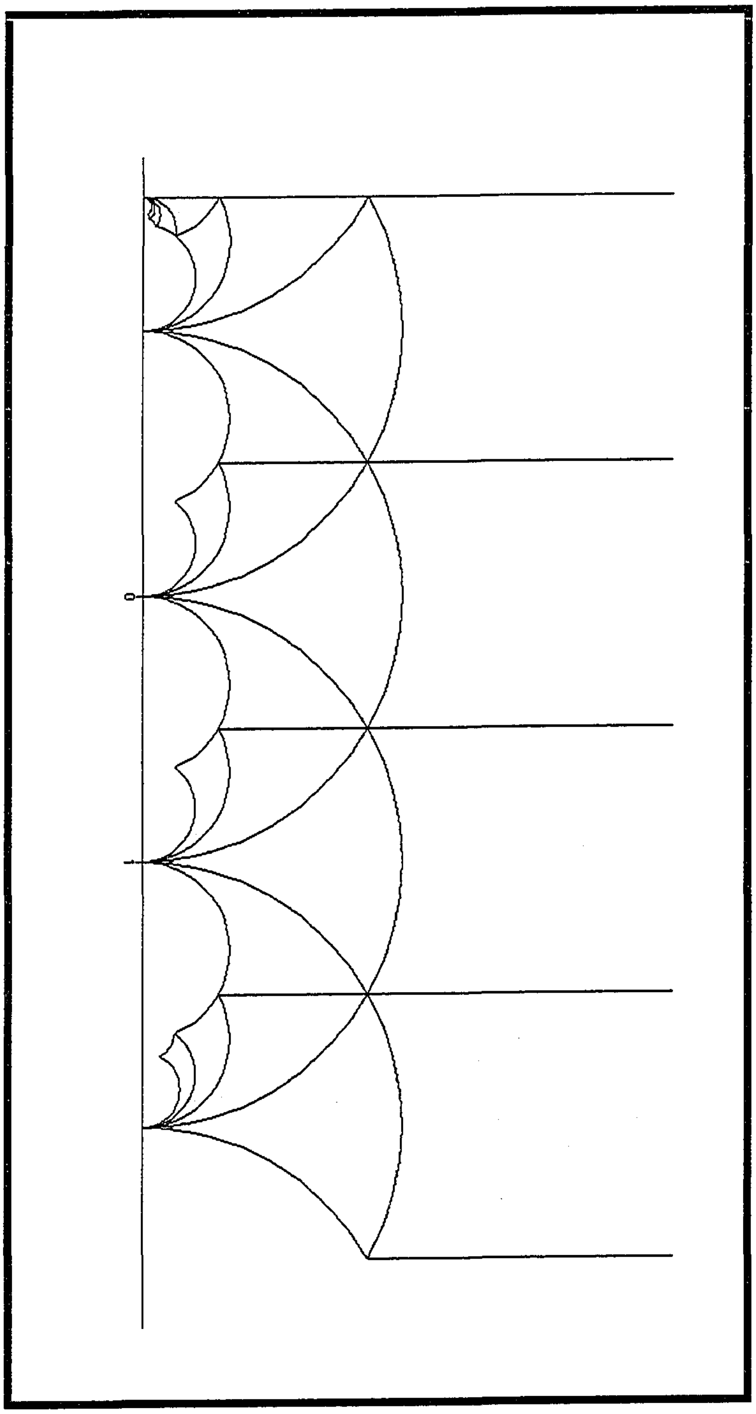


Figure 5.13: H-convex SFD for Γ(4)

Example 5.2 *Consider the normal subgroup* $\Gamma(5)$ *of* $\Gamma(1)$. *We know that*

(i) $[\Gamma(1) : \Gamma(5)] = 60$;

(ii) *The parabolic class number of* $\Gamma(5)$ *is 12.*

An h-convex SFD for $\Gamma(5)$ *is given in Figure 5.14.*

Example 5.3 *Consider the normal subgroup* $\Gamma(6)$ *of* $\Gamma(1)$. *We know that*

(i) $[\Gamma(1) : \Gamma(6)] = 72$;

(ii) *The parabolic class number of* $\Gamma(6)$ *is 12.*

An h-convex SFD for $\Gamma(6)$ *is given in Figure 5.15.*

Example 5.4 *Consider the normal subgroup* $\Gamma(1,1,3)$ *of* $\Gamma(1)$. *We know that*

(i) $[\Gamma(1) : \Gamma(1,1,3)] = 18$;

(ii) *The parabolic class number of* $\Gamma(1,1,3)$ *is 3.*

An h-convex SFD for $\Gamma(1,1,3)$ *is given in Figure 5.16.*

Example 5.5 *Consider the normal subgroup* $\Gamma(2,0,1)$ *of* $\Gamma(1)$. *We know that*

(i) $[\Gamma(1) : \Gamma(2,0,1)] = 24$;

(ii) *The parabolic class number of* $\Gamma(2,0,1)$ *is 4.*

An h-convex SFD for $\Gamma(2,0,1)$ *is given in Figure 5.17.*

Example 5.6 *Consider the normal subgroup* $\Gamma(1,4,7)$ *of* $\Gamma(1)$. *We know that*

(i) $[\Gamma(1) : \Gamma(1,4,7)] = 42$;

(ii) *The parabolic class number of* $\Gamma(1,4,7)$ *is 7.*

An h-convex SFD for $\Gamma(1,4,7)$ *is given in Figure 5.18.*

Example 5.7 *Consider the normal subgroup* $\Gamma(3,0,1)$ *of* $\Gamma(1)$. *We know that*

(i) $[\Gamma(1) : \Gamma(3,0,1)] = 54$;

(ii) *The parabolic class number of* $\Gamma(3,0,1)$ *is 9.*

An h-convex SFD for $\Gamma(3,0,1)$ *is given in Figure 5.19.*

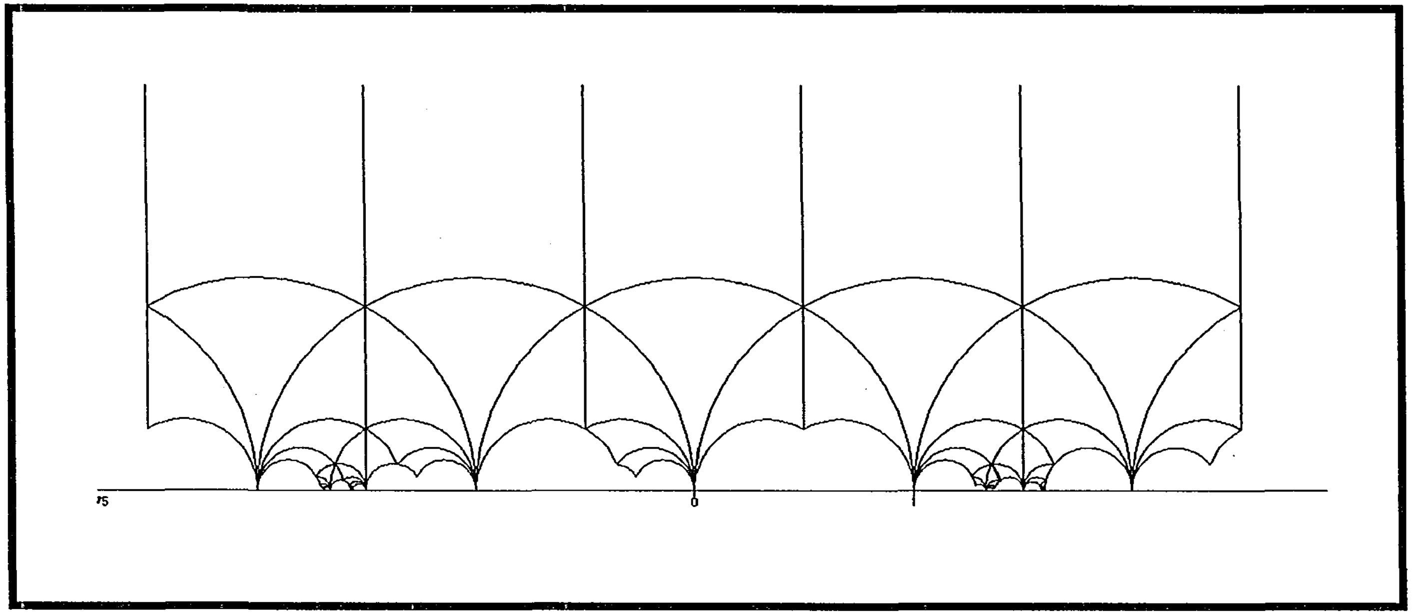


Figure 5.14: H-convex SFD for $\Gamma(5)$

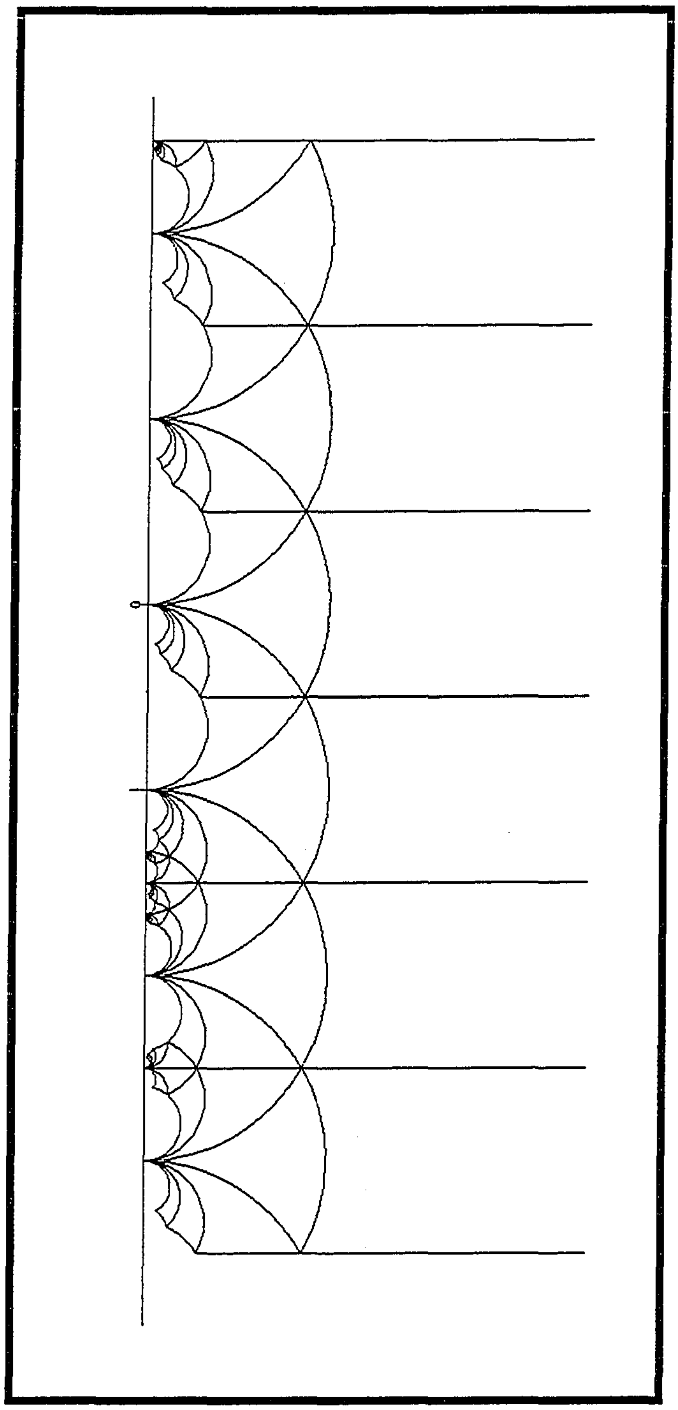


Figure 5.15: H-convex SFD for Γ(6)

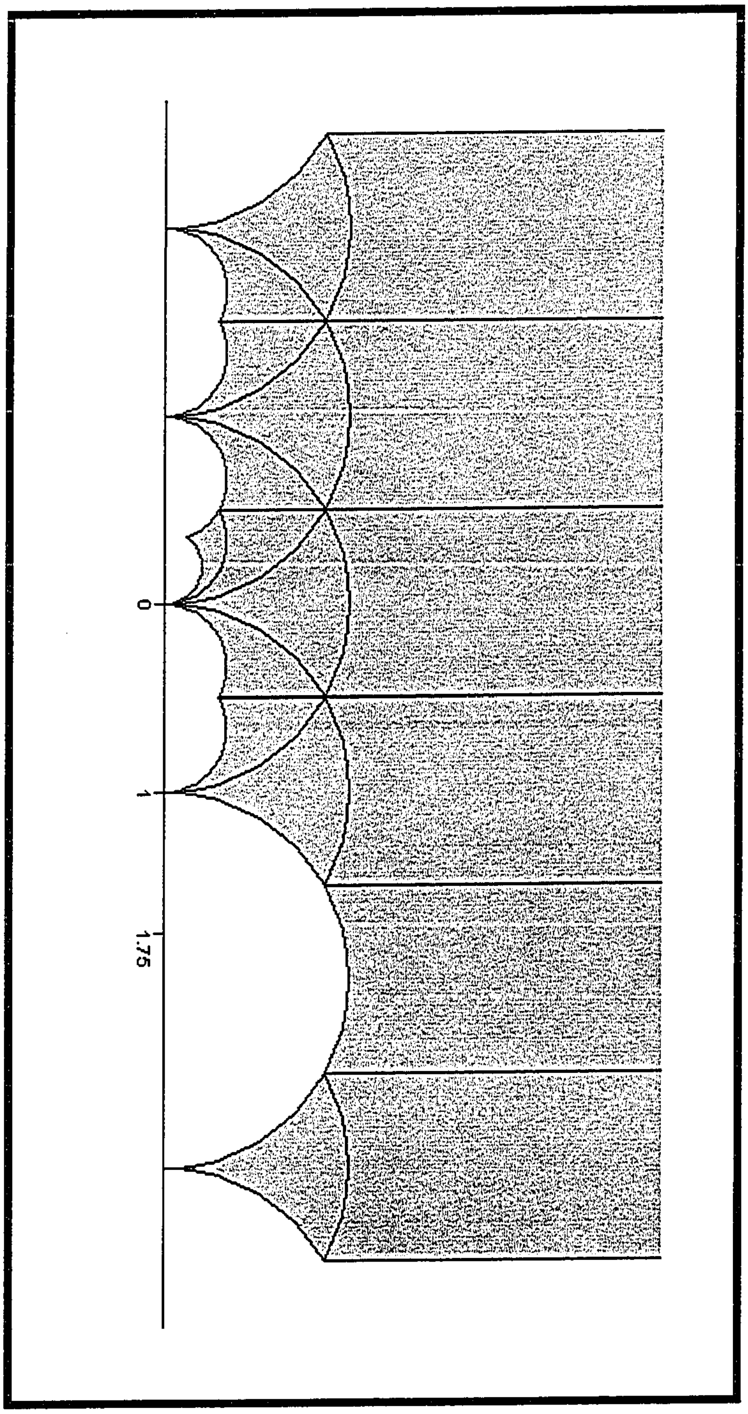


Figure 5.16: SFD for $\Gamma(1, 1, 3)$

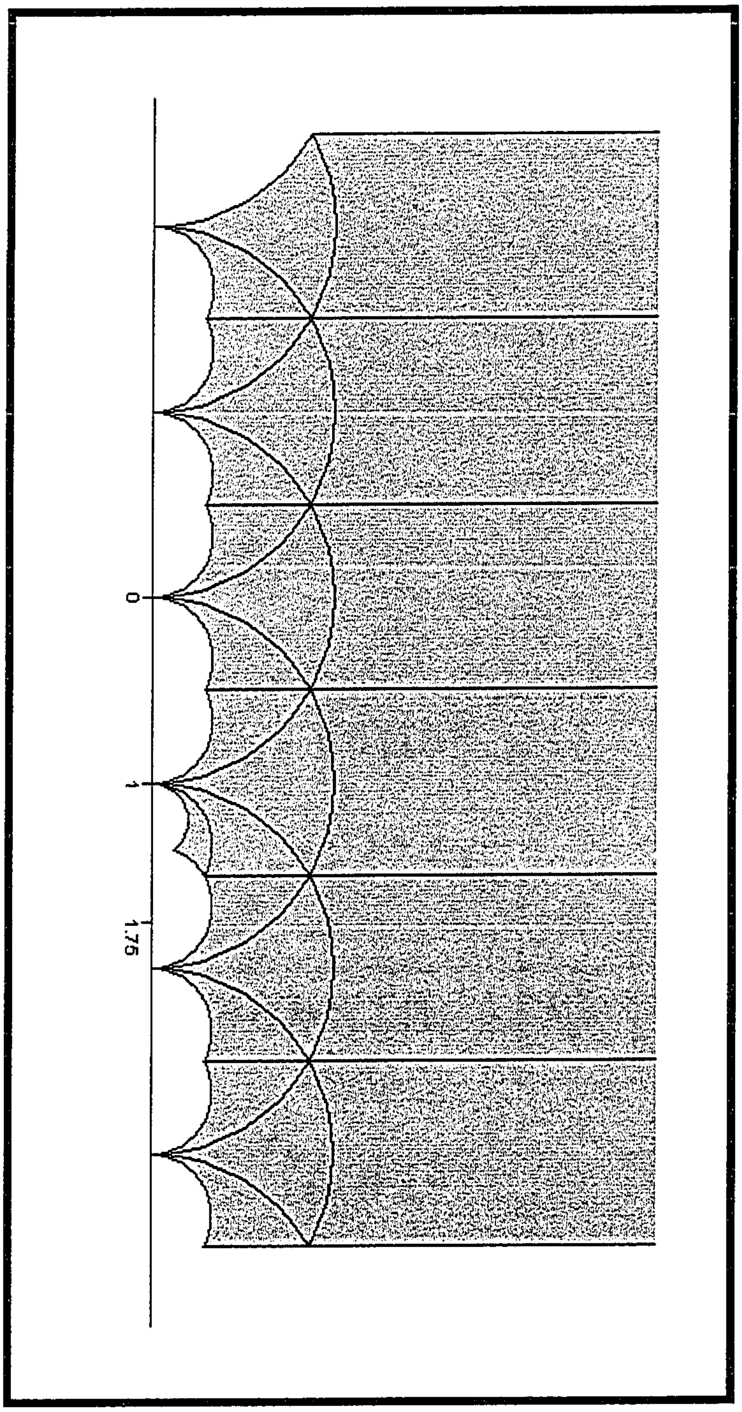


Figure 5.17: SFD for Γ(2, 0, 1)

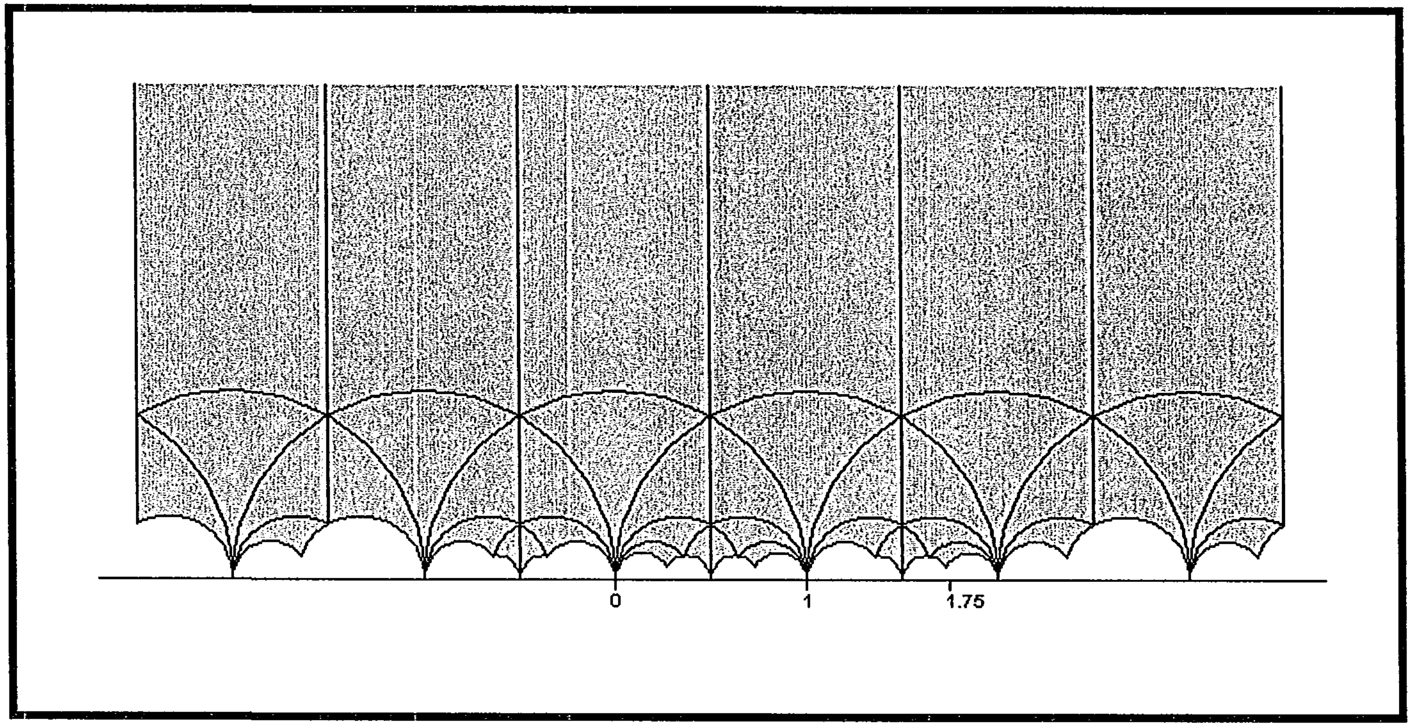


Figure 5.18: SFD for $\Gamma(1,4,7)$

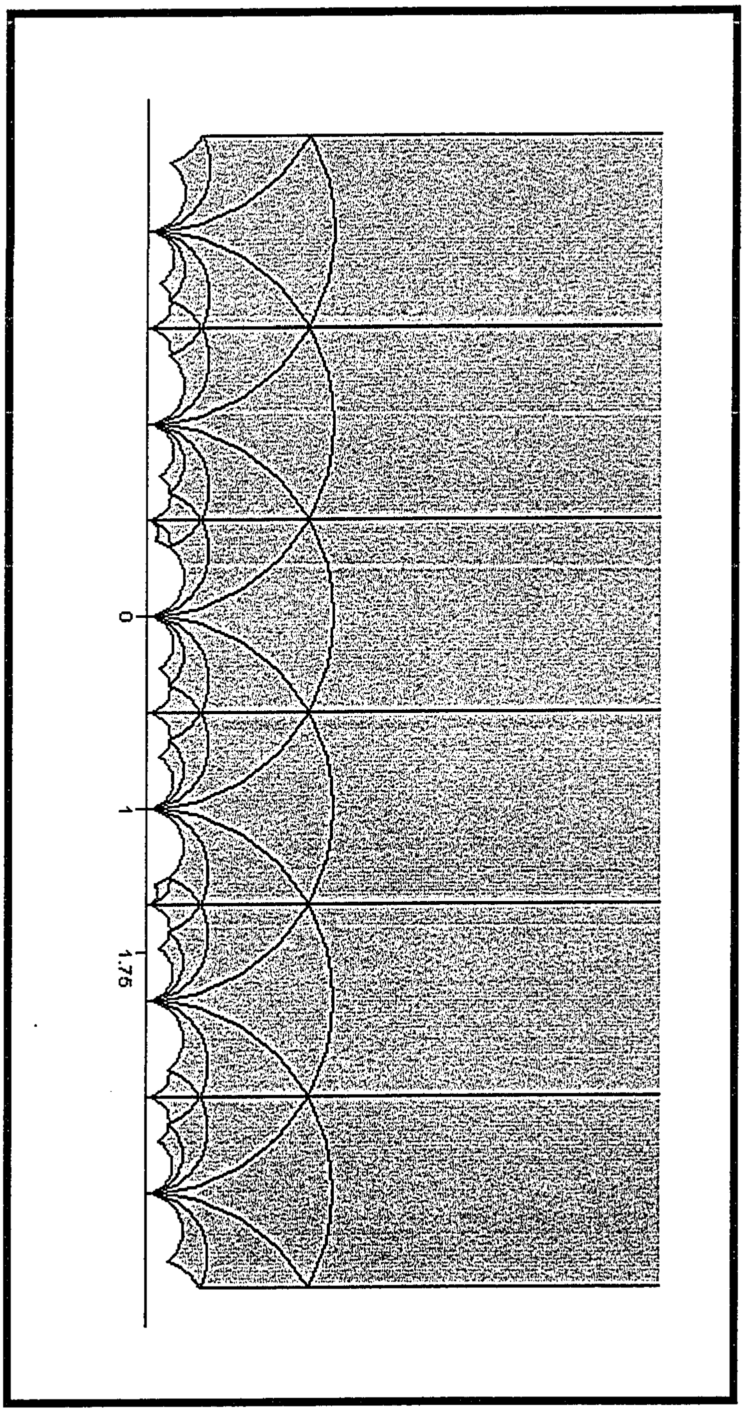


Figure 5.19: H-convex SFD for Γ(3, 0, 1)

5.2 H-convex Fundamental Domain for arbitrary subgroups of finite index of $H(\lambda)$

The interior of the set

$$\mathcal{R}^\lambda = \left\{\tau \in \mathbb{H} : 0 \leq Re(\tau) \leq \frac{\lambda}{2},\ |\tau - \frac{1}{\lambda}| \geq \frac{1}{\lambda}\right\}$$

is a fundamental domain for $H(\lambda)$. It is well known that if G is a subgroup of $H(\lambda)$ of finite index such that $H(\lambda) = G \cdot \Sigma$, and

$$\mathcal{R}_G = \bigcup_{M \in \Sigma} M(\mathcal{R}^\lambda),$$

then the interior of the set $\mathcal{R}_G$ is a fundamental domain for G. For example if $\lambda = \frac{1+\sqrt{5}}{2}$ ($q = 5$), then the shaded region in Figure 5.20 is $\mathcal{R}^\lambda$. Moreover, Figure 5.20 shows the tessellation of the upper half-plane by the fundamental domain of $H\left(\frac{1+\sqrt{5}}{2}\right)$.

In this section we will see the advantage of using $\mathcal{R}^\lambda$ instead of $\mathcal{R}_\lambda$.(In section 4.1 the use of $\mathcal{R}_\lambda$ forces us to consider only normal subgroups of $H(\lambda)$.) Among other things $\mathcal{R}^\lambda$ avoids much of the difficulty we face in using $\mathcal{R}_\lambda$, because it contains only one vertex of order q instead of two. As a result we will be able to show that for any subgroup G of $H(\lambda)$ of finite index we can choose a complete right coset system $\Sigma = \{A_1, \cdots, A_\mu\}$ such that the interior of the set

$$\mathcal{R}_G = \bigcup_{k=1}^{\mu} A_k(\mathcal{R}^\lambda) \tag{5.3}$$

is an h-convex fundamental domain for G. In this section we will give an elementary algorithmic proof of the existence of a system of right cosets $\Sigma \subset H(\lambda)$ such that the interior of the set $\mathcal{R}_G$ is h-convex. We call the image of $\mathcal{R}^\lambda$ by a transformation in $H(\lambda)$ a triangle. Note that the transformations TS_λ and TS_λ^{-1} have finite order and their order is q. Before we state the main theorem we need the following fact. If $H = \langle h \rangle$ is a cyclic group of order m and $K \leqq H$, then there exists a smallest natural number d such that $K = \langle h^d \rangle$ and $H/K = \langle hK \rangle$. One application of the above fact is the following lemma.

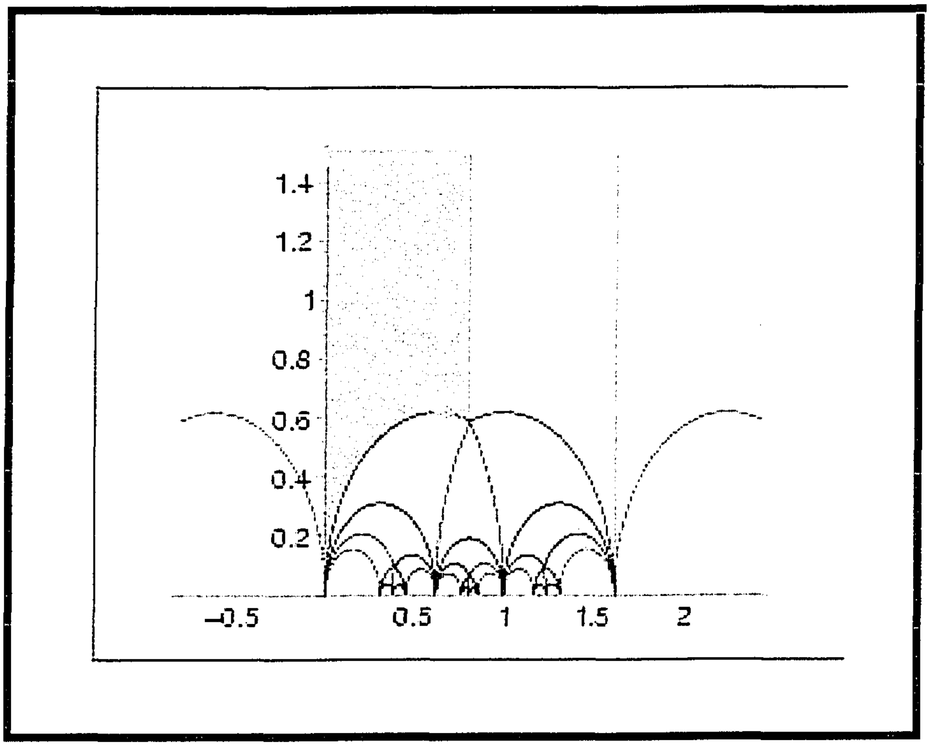


Figure 5.20: Tesselation of $\mathbb{H}$

Lemma 5.2 *Let* $\lambda = 2\cos\left(\frac{\pi}{q}\right)$ *for* $q \in \mathbb{N}$, $q \geq 3$ *and* $\eta_\lambda = e^{\frac{\pi i}{q}}$.

(a) If $\Gamma \leqq H(\lambda)$ *and* $B \in H(\lambda)$, *then there is a smallest positive integer* d *with* $d|q$ *and*

$$\Gamma_{B(\eta_\lambda)} = \left\langle B\left(TS_\lambda^{-1}\right)^d B^{-1}\right\rangle.$$

(b) For any $M \in H(\lambda)$, $M\left(\mathcal{R}^\lambda\right)$ *is adjacent only to* $MT\left(\mathcal{R}^\lambda\right)$, $MS_\lambda T\left(\mathcal{R}^\lambda\right)$, *and* $MTS_\lambda^{-1}\left(\mathcal{R}^\lambda\right)$.

The following lemma is very useful and easy to see. We state it without proof.

Lemma 5.3 *Suppose that* $\Gamma \leqq H(\lambda)$, $[H(\lambda) : \Gamma] = \mu < \infty$, *and* $\emptyset \neq \Sigma \subseteq H(\lambda)$ *with the following properties:*

(i) Any two elements of Σ *are inequivalent with respect to the group* Γ;

(ii) $\mathcal{R} = \bigcup_{A\in\Sigma} A(\mathcal{R}^\lambda)$ *is h-convex.*

If $v \in \mathbb{H}$ is a vertex of $\mathcal{R}$ and $A_1, A_2, \cdots, A_m$ are the elements of Σ such that $A_j(\eta_\lambda) = v$ for all $1 \leq j \leq m$, then

1. *for each $j = 2, \cdots, m$ there exists $n_j \in \mathbb{N}$ such that $A_j = A_1(TS_\lambda^{-1})^{n_j}$ (reindex if necessary);*
2. *$m \leq \frac{q}{2}$.*

Theorem 5.2 *If G is a subgroup of $H(\lambda)$ and $[H(\lambda) : G] = \mu < \infty$, then there exists a set Σ such that*

(a) $H(\lambda) = G \cdot \Sigma$;

(b) the interior of the set $\mathcal{R}_\Sigma = \bigcup_{M\in\Sigma} M(\mathcal{R}^\lambda)$ is an h-convex fundamental domain for G.

Algorithmic Proof: We will construct an increasing sequence of sets,

$$\Sigma_1 \subset \Sigma_2 \subset \Sigma_3 \subset \ldots \subset \Sigma_m = \Sigma,$$

where m is at most the index $[H(\lambda) : G]$. At step k we adjoin at least one and at most q elements of $H(\lambda)$ to Σ_{k-1}.

STEP 1

Let $M_1 \in G$. We may always take $M_1 = I$, if we choose. Then there exists a smallest positive integer d_1 such that

$$G_{M_1(\eta_\lambda)} = \langle M_1(TS_\lambda^{-1})^{d_1} M_1^{-1} \rangle .$$

Now we let $\Sigma_1 = \{M_1, M_1(TS_\lambda^{-1}), \cdots M_1(TS_\lambda^{-1})^{d_1-1}\}$. If $d_1 = q$, then Σ_1 contains q elements, otherwise Σ_1 contains at most $\frac{q}{2}$ elements. Therefore the set

$$\mathcal{R}_1 = \bigcup_{A\in\Sigma_1} A(\mathcal{R}^\lambda)$$

is a closed connected set.

We want to show that $\mathcal{R}_1$ is h-convex. If $d_1 = q$, then $\mathcal{R}_1$ has no vertex in $\mathbb{H}$ and hence it is locally h-convex. Therefore by Theorem 3.4 $\mathcal{R}_1$ is h-convex. If $d_1 < q$, then the only vertex of $\mathcal{R}_1$ in $\mathbb{H}$ is $v_1 = M_1(\eta_\lambda)$ and the interior angle at v_1 is $\left(\frac{2\pi}{q}\right) d_1$ which is at most $\left(\frac{q}{2}\right)\left(\frac{2\pi}{q}\right) = \pi$. Therefore $\mathcal{R}_1$ is locally h-convex and hence by Theorem 3.4 it is h-convex.

Note that for any $0 \le t \le q - d_1$, we have

$$\begin{aligned} M_1(TS_\lambda^{-1})^{d_1+t} &= M_1(TS_\lambda^{-1})^{d_1+ad_1+t_0} \\ &= \underbrace{M_1(TS_\lambda^{-1})^{d_1(1+a)}M_1^{-1}}_{\in G}\underbrace{M_1(TS_\lambda^{-1})^{t_0}}_{\in\Sigma}, \end{aligned}$$

where $t = ad_1 + t_0$ with $0 \le t_0 \le d_1 - 1$. Hence

$$M_1(TS_\lambda^{-1})^{d_1+t} \overset{G}{\sim} \Sigma_1, \quad \forall t,\ 0 \le t \le q - d_1.$$

Terminate the process if either $[H(\lambda) : G] = |\Sigma_1|$ or there is no $B \in H(\lambda)$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $\mathcal{R}_1$ and $B \overset{G}{\nsim} \Sigma_1$.(These two statements are equivalent by Corollary 4.1.) Otherwise GO TO STEP 2.

STEP 2

There exists $B \in H(\lambda)$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $\mathcal{R}_1$ and $B \overset{G}{\nsim} \Sigma_1$. Then there exists $A_1 \in \Sigma_1$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $A_1(\mathcal{R}^\lambda)$. By Lemma 5.2, $B = A_1T$ or $B = A_1TS_\lambda^{-1}$ or $B = A_1(TS_\lambda^{-1})^{q-1}$. Clearly $B = A_1T$, because $A_1TS_\lambda^{-1} \overset{G}{\sim} \Sigma_1$ and $A_1(TS_\lambda^{-1})^{q-1} \overset{G}{\sim} \Sigma_1$. That means $B(\mathcal{R}^\lambda)$ and $A_1(\mathcal{R}^\lambda)$ share the common side $B(\mathcal{I})$, where $\mathcal{I}$ is the positive imaginary axis. By Lemma 5.2 there exists d_2, the smallest positive integer such that

$$G_{B(\eta_\lambda)} = \left\langle B\left(TS_\lambda^{-1}\right)^{d_2} B^{-1}\right\rangle.$$

First we want to show that for each $0 \le t \le d_2 - 1$, $B(TS_\lambda^{-1})^t \overset{G}{\nsim} \Sigma_1$. Suppose that there exist $0 \le t \le d_2 - 1$ such that $B(TS_\lambda^{-1})^t \overset{G}{\sim} \Sigma_1$. Then there exist $M \in G$ and $0 \le a \le d_1 - 1$ such that

$$B(TS_\lambda^{-1})^t = MM_1(TS_\lambda^{-1})^a,$$

$$
\begin{aligned}
B &= (MM_1)(TS_\lambda^{-1})^{a+q-t} \\
&= \underbrace{(MM_1)(TS_\lambda^{-1})^{bd_1}M_1^{-1}}_{\in G}\underbrace{M_1(TS_\lambda^{-1})^{t_0}}_{\in \Sigma_1},
\end{aligned}
$$

where $a+q-t = bd_1 + t_0$, $0 \le t_0 \le d_1 - 1$. This is a contradiction to $B \overset{G}{\not\sim} \Sigma_1$. Now we let $\Sigma_2 = \Sigma_1 \cup \{B, B(TS_\lambda^{-1}), \cdots, B(TS_\lambda^{-1})^{d_2-1}\}$ and $\mathcal{R}_2 = \bigcup_{A\in\Sigma_2} A(\mathcal{R}^\lambda)$. Then

(i) $v_2 = B(\eta_\lambda) \neq v_1 = M_1(\eta_\lambda)$;

(ii) a vertex of $\mathcal{R}_1$ which lies in $\mathbb{H}$ is also a vertex of $\mathcal{R}_2$ with the same interior angle;

(iii) if $d_2 < q$, then v_2 is the only additional vertex of $\mathcal{R}_2$ which lies in $\mathbb{H}$ and the interior angle at vertex v_2 (of $\mathcal{R}_2$) is $\frac{2\pi}{q}d_2 \le \pi$;

(iv) if $d_2 = q$, then v_2 is an interior point of $\mathcal{R}_2$;

(v) $\mathcal{R}_2$ is closed and connected.

Therefore $\mathcal{R}_2$ is locally h-convex and connected and hence by Theorem 3.4, $\mathcal{R}_2$ is h-convex.

Terminate the process, if either $[H(\lambda) : G] = |\Sigma_2|$ or there is no $B \in H(\lambda)$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $\mathcal{R}_2$ and $B \overset{G}{\not\sim} \Sigma_2$. Otherwise GO TO STEP 3.

STEP k

There exists $B \in H(\lambda)$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $\mathcal{R}_{k-1}$ and $B \overset{G}{\not\sim} \Sigma_{k-1}$. Then there exists $A_1 \in \Sigma_{k-1}$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $A_1(\mathcal{R}^\lambda)$. By Lemma 5.2, $B = A_1T$ or $B = A_1TS_\lambda^{-1}$ or $B = A_1(TS_\lambda^{-1})^{q-1}$. As before we can easily show that $B = A_1T$. That means $B(\mathcal{R}^\lambda)$ and $A_1(\mathcal{R}^\lambda)$ share the common side $B(\mathcal{I})$. By Lemma 5.2 there exists a smallest positive integer d_k such that

$$G_{B(\eta_\lambda)} = \left\langle B\left(TS_\lambda^{-1}\right)^{d_k} B^{-1}\right\rangle.$$

As we saw in STEP 2, $B(TS_\lambda^{-1})^t \overset{G}{\not\sim} \Sigma_{k-1}$ for any $0 \le t \le d_k - 1$. Now we let $\Sigma_k = \Sigma_{k-1} \cup \{B, B(TS_\lambda^{-1}), \cdots, B(TS_\lambda^{-1})^{d_k-1}\}$ and $\mathcal{R}_k = \bigcup_{A\in\Sigma_k} A(\mathcal{R}^\lambda)$. Then

(i) every vertex of $\mathcal{R}_{k-1}$ in $\mathbb{H}$ is a vertex of $\mathcal{R}_k$;

(ii) $\mathcal{R}_k$ contains one additional vertex $v_k = B(\eta_\lambda)$ in $\mathbb{H}$, if $d_k < q$;

(iii) if $v_r \in \mathbb{H}$ is a vertex of $\mathcal{R}_k$, for some $r = 1, 2, \cdots, k$, and θ_r is its interior angle, then $d_r \le \frac{q}{2}$ and $\theta_r = \left(\frac{2\pi}{q}\right) d_r \le \pi$;

(iv) $\mathcal{R}_k$ is closed and connected.

Therefore $\mathcal{R}_k$ is locally h-convex and connected and hence by Theorem 3.4 $\mathcal{R}_k$ is h-convex.

Terminate the process, if either $[H(\lambda) : G] = |\Sigma_k|$ or there is no $B \in H(\lambda)$ such that $B(\mathcal{R}^\lambda)$ is adjacent to $\mathcal{R}_k$ and $B \overset{G}{\not\sim} \Sigma_k$. Otherwise GO TO THE NEXT STEP.

The process terminates after a finite number of steps, because $[H(\lambda) : G] = \mu < \infty$. Suppose the process terminates at STEP m. At the end of STEP m we get

$$\Sigma_m = \{M_1, M_1(TS_\lambda^{-1}), \cdots, M_1(TS_\lambda^{-1})^{d_1-1}, ..., M_m, M_m(TS_\lambda^{-1}), \cdots, M_m(TS_\lambda^{-1})^{d_m-1}\}$$

and $\mathcal{R}_m = \bigcup_{A\in\Sigma_m} A(\mathcal{R}^\lambda)$, which is h-convex, where either

$$|\Sigma_m| = [H(\lambda) : G]$$

or

$$B(\mathcal{R}^\lambda) \text{ is adjacent to } \mathcal{R}_m \quad \Rightarrow \quad B \overset{G}{\sim} \Sigma_m.$$

These two statements are equivalent by Corollary 4.1.

Q.E.D

5.3 Examples

Example 5.8 $\Gamma(1,1,3)$

Note

- $[\Gamma(1):\Gamma(1,1,3)] = 18$
- The set Σ containing the transformations $I, T, TS^{-1}, ST, S^{-1}, TST, TS^{-1}T, S, S^{-1}T, TS, TS^{-2}, TS^{-2}TS^{-1}, TS^{-1}TS^{-2}, S^2T, S^{-2}, TSTS^2T, TSTS^{-1}, TS^2T$ is a complete right coset system of $\Gamma(1)$ modulo $\Gamma(1,1,3)$;
- $\mathcal{R} = \{\overline{\bigcup_{A\in\Sigma} A(\mathcal{R}^0)}\}^o$, shown in Figure 5.21, is an h-convex fundamental domain for $\Gamma(1,1,3)$.
- The rank of $\Gamma(1,1,3)$ is 4 and the generators of $\Gamma(1,1,3)$ are:
  $M_1 = S^{-2}TS^2T$
  $M_2 = TSTS^3TS^{-1}$
  $M_3 = TSTS^{-2}TST$
  $M_4 = TS^2TS^{-2}$

Example 5.9 $\Gamma(2,0,1)$

- $[\Gamma(1):\Gamma(2,0,1)] = 24$
- The set Σ containing the transformations $TS^{-3}TS^{-1}, I, S^{-2}T, S^{-2}, TSTS^2T, S^2T, TS^{-1}TS^{-2}, S^{-1}T, ST, T, TS^{-1}, S^{-1}, TS, TST, TS^{-2}T, TS^{-3}, TS^2T, TSTS^{-1}, TS^{-1}T, S^{-3}, S^{-2}TST, S, TS^{-2}TS^{-1}, TS^{-2}$ is a complete right coset system of $\Gamma(1)$ modulo $\Gamma(2,0,1)$;
- $\mathcal{R} = \{\overline{\bigcup_{A\in\Sigma} A(\mathcal{R}^0)}\}^o$, shown in Figure 5.22, is an h-convex fundamental domain for $\Gamma(1,1,3)$.
- The rank of $\Gamma(2,0,1)$ is 5 and the generators of $\Gamma(2,0,1)$ are:
  $M_1 = S^{-3}TS^3T$
  $M_2 = S^{-2}TSTS^{-2}T$

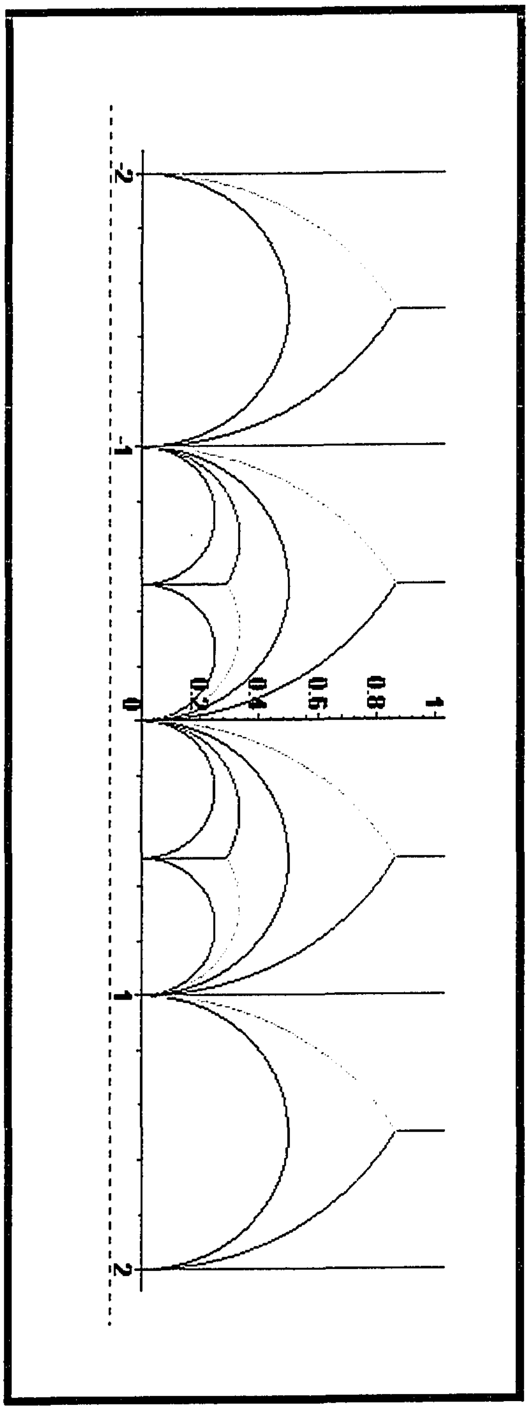


Figure 5.21: h-convex SFD for $\Gamma(1, 1, 3)$

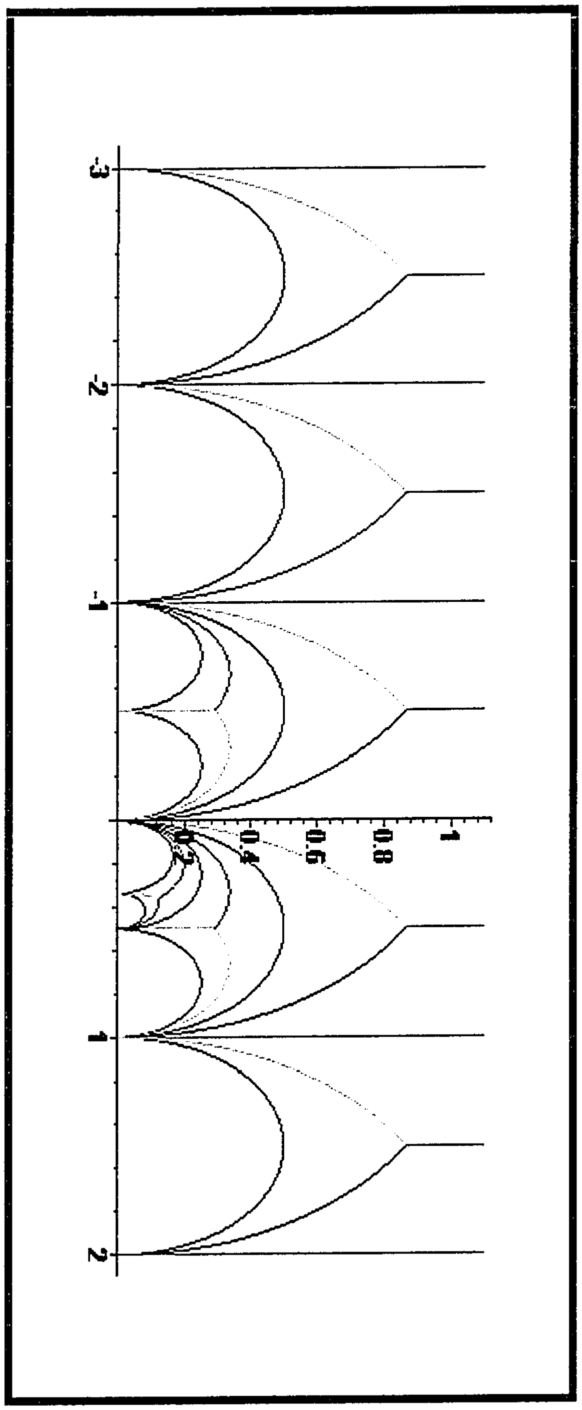


Figure 5.22: h-convex SFD for Γ(2, 0, 1)

$$
\begin{aligned}
M_3 &= TSTS^2TS^{-1}TST \\
M_4 &= TS^2TS^2TS^{-1} \\
M_5 &= TS^{-3}TS^{-3}
\end{aligned}
$$

Example 5.10 $\Gamma(1,2,7)$ and $\Gamma(1,4,7)$

- $[\Gamma(1):\Gamma(1,2,7)] \;=\; 42 \;=\; [\Gamma(1):\Gamma(1,4,7)]$
- The set Σ containing the transformations $I, T, TS^{-1}, ST, S^{-1}, TST, TS^{-1}T, S, S^{-1}T, TS, TS^{-2}, TS^{-2}TS^{-1}, TS^{-1}TS^{-2}, S^2T, S^{-2}, TSTS^2T, TSTS^{-1}, TS^2T, TS^{-2}T, TS^{-1}TS, TS^{-1}TS^{-2}T, S^2, S^{-2}T, TSTS^2, TS^2TS, TS^2, TS^{-3}, TS^{-3}TS^{-1}, TS^{-2}TS^{-2}, TS^{-1}TS^2T, TS^{-1}TS^{-3}, TS^{-1}TS^{-2}TST, S^2TS^{-1}, S^3T, S^{-3}, S^{-2}TST, TSTS^2TS^{-1}, TSTS^3T, TSTS^{-2}, TS^2TS^2T, TS^2TS^{-1}, TS^3T$ is a complete right coset system of $\Gamma(1)$ modulo $\Gamma(1,m,7)$ where $m \in \{2,4\}$.
- $\mathcal{R} \;=\; \{\overline{\bigcup_{A\in\Sigma} A(\mathcal{R}^0)}\}^o$, shown in Figure 5.23, is an h-convex fundamental domain for $\Gamma(1,2,7)$ and also $\Gamma(1,4,7)$.
- The rank of $\Gamma(1,2,7)$ and $\Gamma(1,4,7)$ is 8 and

  (a). Generators of $\Gamma(1,2,7)$ are:

$$
\begin{aligned}
M_1 &= S^{-6} \\
M_2 &= S^{-2}TSTS^{-1}TS^2T \\
M_3 &= TSTS^2TS^{-1}TS^2TS^{-1} \\
M_4 &= TSTS^3TS^{-2}TST \\
M_5 &= TSTS^{-2}TSTS^{-1}TST \\
M_6 &= TS^2TS^3TS^{-2} \\
M_7 &= TS^2TS^{-2}TSTS^{-1} \\
M_8 &= TS^6T
\end{aligned}
$$

  (b). Generators of $\Gamma(1,4,7)$ are:

$$
\begin{aligned}
N_1 &= S^{-6} \\
N_2 &= S^{-2}TS^2TS^{-1}TST
\end{aligned}
$$

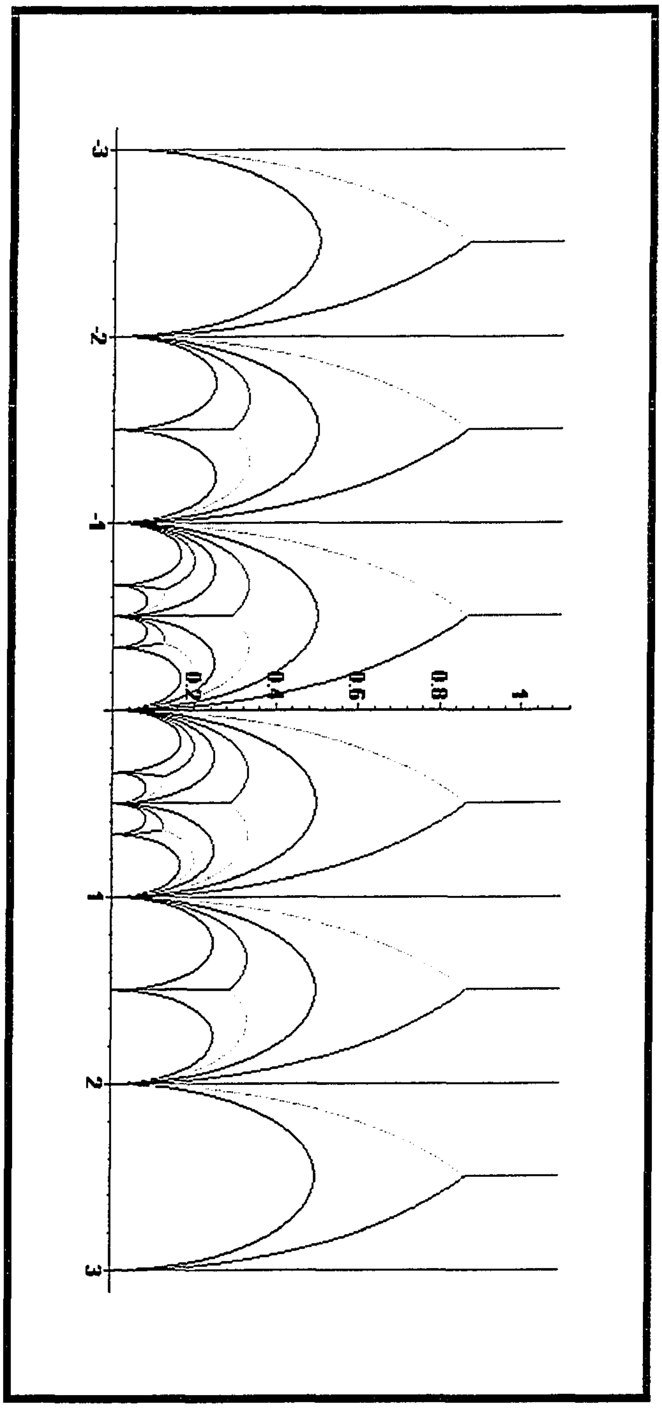


Figure 5.23: h-convex SFD for Γ(1, 2, 7) and Γ(1, 4, 7)

$$N_3 = TSTS^2TS^{-2}TS^2T$$
$$N_4 = TSTS^3TS^{-1}TSTS^{-1}$$
$$N_5 = TSTS^{-2}TS^2TS^{-1}$$
$$N_6 = TS^2TS^2TS^{-2}TST$$
$$N_7 = TS^3TS^2TS^{-2}$$
$$N_8 = TS^6T$$

- We can easily show that $\Gamma(1,2,7) \cap \Gamma(1,4,7) = \Gamma(7,4,7)$[Use the definition].

Example 5.11 $\Gamma(3,0,1)$

- $[\Gamma(1) : \Gamma(3,0,1)] = 54$

- The set Σ containing the transformations $S^2, TSTS^2T, TS^3TS^2T, TS^3, T$ $TS^2TS, TS^{-3}, TS^2TS^{-1}, TS^{-1}TS, TS^2, TS^{-3}TS^{-1}, S^{-1}T, ST, T, TS^{-1}$, $S^{-1}, TST, TSTS^2, TS^{-1}TS^{-2}, S^{-2}, S^2T, TS^2TS^2T, TSTS^{-2}, TSTS^2TS^{-1}$, $TSTS^3T, TS^{-1}TS^2T, TS^{-2}TS^{-2}, TSTS^{-1}, TS^2T, TS^{-1}T, S^{-2}T, S^{-3}$, $S^{-2}TST, TSTS^3, S^3T, TS^{-2}, TS^{-2}TS^{-1}, TS^3TS, TS^{-2}T$, $S^2TS^{-1}, S, TS^{-1}TS^2, S^2TS^{-1}T, TSTS^4T, TSTS^3TS^{-1}$, $TS^{-1}TS^{-2}TST, TS^{-1}TS^{-3}, TS, TS^{-1}TS^{-2}T, TS^{-1}TS^3T, TS^{-1}TS^2TS^{-1}$, $S^2TS^{-1}TST, S^2TS^{-2}, TS^2TS^{-2}$ is a complete right coset system of $\Gamma(1)$ modulo $\Gamma(3,0,1)$;

- $\mathcal{R} = \{\overline{\bigcup_{A\in\Sigma} A(\mathcal{R}^0)}\}^o$, shown in Figure 5.24, is an h-convex fundamental domain for $\Gamma(3,0,1)$.

- The rank of $\Gamma(3,0,1)$ is 10 and the generators of $\Gamma(3,0,1)$ are :

$$M_1 = S^{-6}$$
$$M_2 = S^{-2}TSTS^{-1}TSTS^{-2}T$$
$$M_3 = TSTS^2TS^{-1}TSTS^{-1}TST$$
$$M_4 = TSTS^3TS^{-1}TS^2TS^{-2}$$
$$M_5 = TSTS^{-5}TS$$
$$M_6 = TS^2TS^2TS^{-1}TSTS^{-1}$$

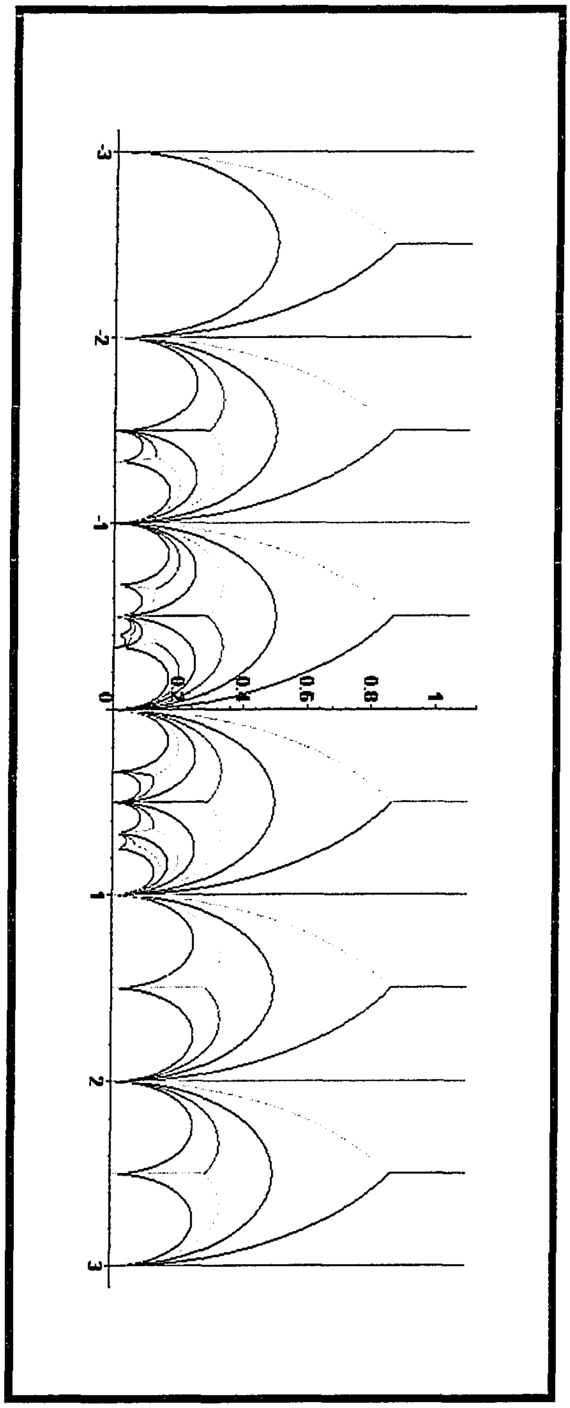


Figure 5.24: h-convex SFD for Γ(3,0,1)

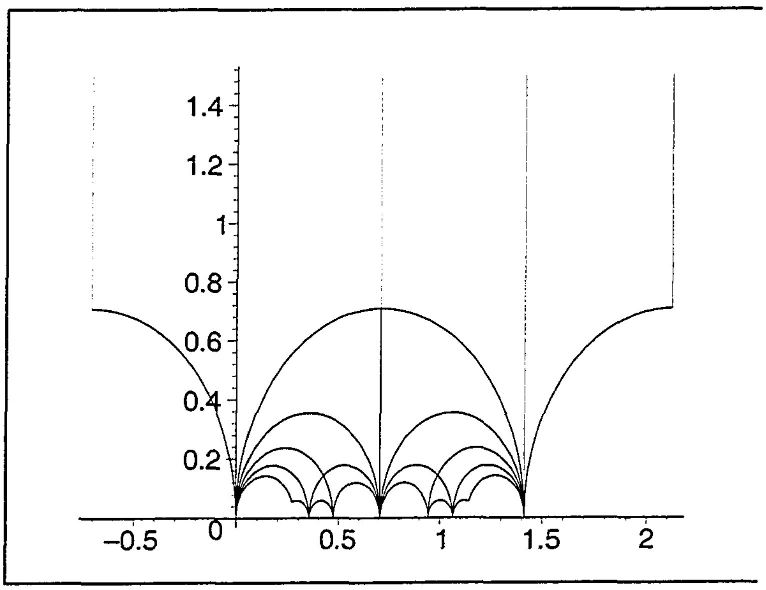
$$
\begin{aligned}
M_7 &= TS^2TS^{-2}TSTS^{-2}TST \\
M_8 &= TS^6T \\
M_9 &= TS^{-3}TS^{-2}TSTS^{-2} \\
M_{10} &= TS^{-1}TS^{-6}TST.
\end{aligned}
$$

Consider the group $H(\sqrt{2})$. It is well known ([9],[30]) that $H(\sqrt{2})$ consists of the mappings of all of the following types:

(i) $N(\tau) = \frac{a\tau + b\sqrt{2}}{c\sqrt{2}\tau + d}$, $a, b, c, d \in \mathbb{Z}, ad - 2bc = 1,$

(ii) $N(\tau) = \frac{a\sqrt{2}\tau + b}{c\tau + d\sqrt{2}}$, $a, b, c, d \in \mathbb{Z}, 2ad - bc = 1.$

The group $H(\sqrt{2})$ tessellates the upper half-plane as shown below.



Let $n \in \mathbb{N}$. Define

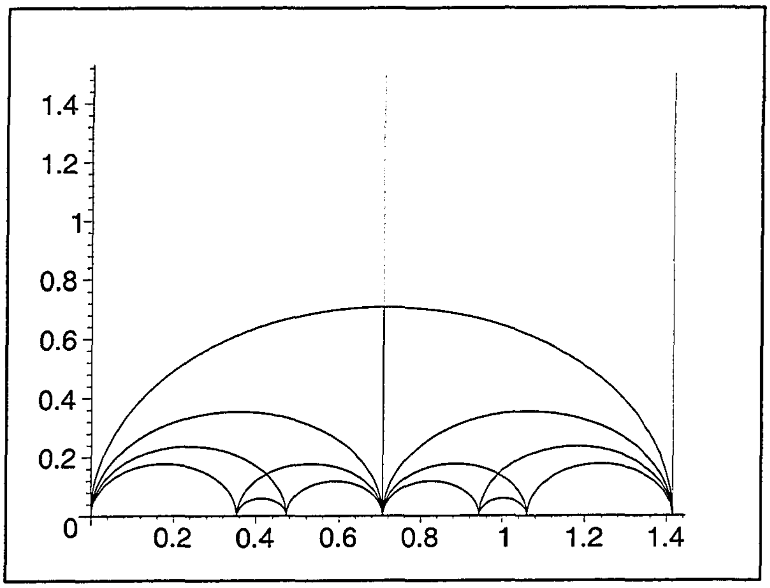
$$H_0^{\sqrt{2}}(n) := \left\{ M \in H(\sqrt{2}) : c \equiv 0(\bmod n) \right\}.$$

Then $H_0^{\sqrt{2}}(n)$ is a subgroup of $H(\sqrt{2})$. It is well known [12] that

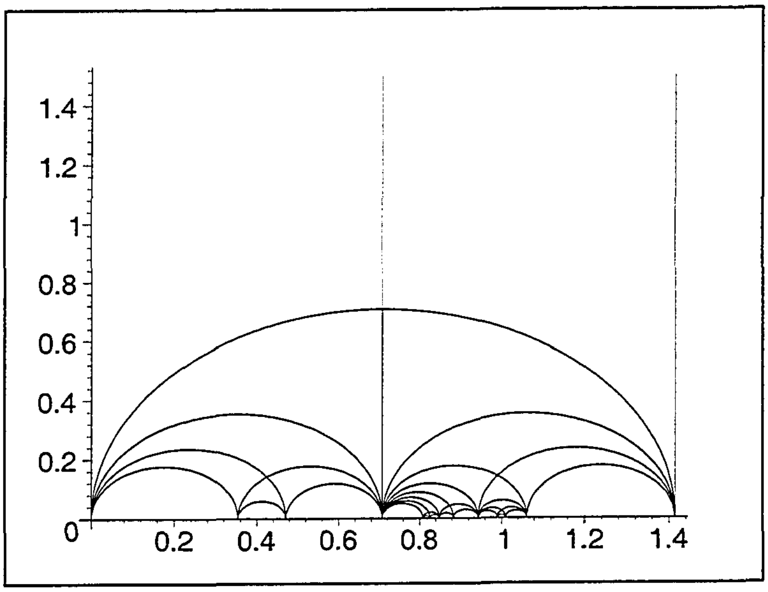
$$\left[H(\sqrt{2}) : H_0^{\sqrt{2}}(n) \right] = \begin{cases} n \prod_{p|n} \left(1 + \frac{1}{p}\right), & \text{if } (2, n) = 1 \\ 2n \prod_{p|n, p \neq 2} \left(1 + \frac{1}{p}\right), & \text{if } 2|n. \end{cases} \tag{5.4}$$

Using the above Theorem we can generate a fundamental domain for any congruence subgroup of the form $H_0^{\sqrt{2}}(n), \;\; n \in \mathbb{N}$.

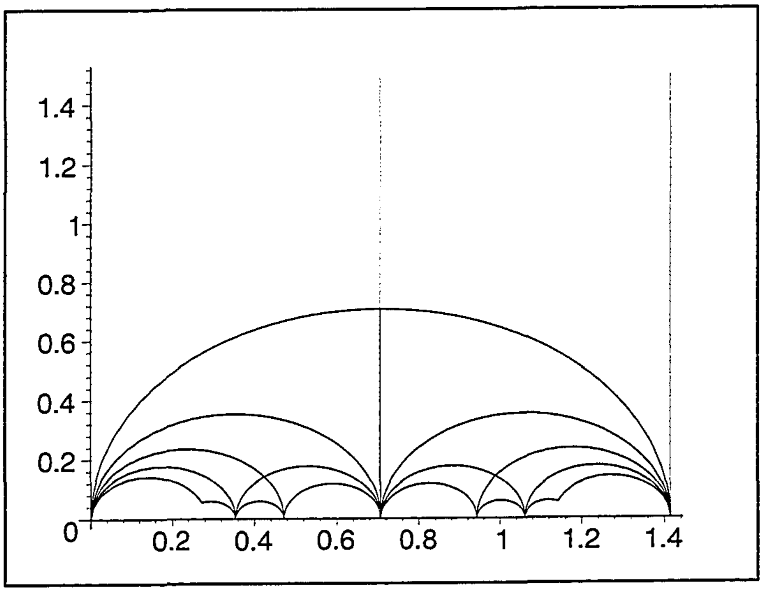
Example 5.12 An h-convex fundamental domain for the subgroup $H_0^{\sqrt{2}}(9)$ is given by the following picture.



Example 5.13 An h-convex fundamental domain for the subgroup $H_0^{\sqrt{2}}(10)$ is given by the following picture.



Example 5.14 An h-convex fundamental domain for the subgroup $H_0^{\sqrt{2}}(13)$ is given by the following picture.



5.4 Generalized Farey Sequence

The ordinary Farey sequence is constructed as follows: start with the first row

$$\frac{0}{1} \quad \frac{1}{1}$$

For the n^{th} row, $n = 2, 3, 4, 5, ...$, we use the rule: form the n^{th} row by copying the $(n-1)^{st}$ row in order and insert the fraction $\frac{a+b}{c+d}$ between the consecutive fractions $\frac{a}{c}$ and $\frac{b}{d}$ of the $(n-1)^{st}$ row. The first four rows of the Farey sequences are as shown below:

$$\begin{array}{ccccccccc} \frac{0}{1} & & & & & & & & \frac{1}{1} \\ \frac{0}{1} & & & & \frac{1}{2} & & & & \frac{1}{1} \\ \frac{0}{1} & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & \frac{1}{1} \\ \frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1} \end{array}$$

It was proved by Cauchy around 1816, after the sequence was constructed by the mineralogist Farey, that if $\frac{a}{c}$ and $\frac{b}{d}$ are two consecutive fractions in the n^{th} row of the Farey sequence, say with $\frac{b}{d}$ to the left of $\frac{a}{c}$, then $ad - bc = 1$. From this we can also conclude that all the fractions appearing in the Farey sequence are in reduced form and in each row the size of the fractions increases from left to right. There is also another way of constructing the Farey sequence from the tessellation of the upper half-plane, using $\mathcal{R}^0$ as shown below.

Suppose that $\lambda = 2cos\left(\frac{\pi}{q}\right)$ for $q = 3, 4, 5, ...$ and let $\theta = \frac{\pi}{q}$. We construct a sequence, which I call the λ-Farey Sequence, as follows: Start with the first row having $q - 1$ entries:

$$\frac{0}{1}\ \frac{\frac{\sin\theta}{\sin\theta}}{\frac{\sin 2\theta}{\sin\theta}}\ \frac{\frac{\sin 2\theta}{\sin\theta}}{\frac{\sin 3\theta}{\sin\theta}}, \ldots, \frac{\frac{\sin(q-3)\theta}{\sin\theta}}{\frac{\sin(q-2)\theta}{\sin\theta}}, \frac{\frac{\sin(q-2)\theta}{\sin\theta}}{\frac{\sin(q-1)\theta}{\sin\theta}} = \frac{\lambda}{1}.$$

For the n^{th} row, n=2,3,4,..., we use the rule: form the n^{th} row by copying the $(n-1)^{st}$ row in order and inserting the following, $q - 2$, λ-fractions between the consecutive fractions $\frac{a}{c}$ and $\frac{b}{d}$, where $\frac{b}{d}$ is to the left of $\frac{a}{c}$, of the $(n-1)^{st}$ row

$$\frac{\frac{a\sin\theta+b\sin 2\theta}{\sin\theta}}{\frac{c\sin\theta+d\sin 2\theta}{\sin\theta}}, \frac{\frac{a\sin 2\theta+b\sin 3\theta}{\sin\theta}}{\frac{c\sin 2\theta+d\sin 3\theta}{\sin\theta}}, \ldots, \frac{\frac{a\sin(q-2)\theta+b\sin(q-1)\theta}{\sin\theta}}{\frac{c\sin(q-2)\theta+d\sin(q-1)\theta}{\sin\theta}}$$

Theorem 5.3 *If $\frac{a}{c}$ and $\frac{b}{d}$ are two consecutive λ-Farey fractions in the n^{th} row, where $\frac{b}{d}$ is to the left of $\frac{a}{c}$, then $ad-bc=1$.*

PROOF: First let us show that the statement is true for n=1. Any two consecutive λ-Farey fractions of the first row are of the type

$$\frac{\frac{\sin m\theta}{\sin\theta}}{\frac{\sin(m+1)\theta}{\sin\theta}}, \frac{\frac{\sin(m+1)\theta}{\sin\theta}}{\frac{\sin(m+2)\theta}{\sin\theta}},$$

for $m=0,1,...,q-3$. Since

$$\begin{aligned}\sin(k+1)\theta\sin(k-1)\theta &= (\sin k\theta cos\theta+\sin\theta cosk\theta)(\sin k\theta cos\theta-\sin\theta cosk\theta)\\ &= \sin^2 k\theta cos^2\theta - \sin^2\theta cos^2 k\theta\\ &= \sin^2 k\theta cos^2\theta - \sin^2\theta\left(1-\sin^2 k\theta\right)\\ &= \sin^2 k\theta - \sin^2\theta,\end{aligned}$$

we have

$$\frac{\sin^2(m+1)\theta-\sin m\theta\sin(m+2)\theta}{\sin^2\theta} = \frac{\sin^2(m+1)\theta-\left(\sin^2(m+1)\theta - \sin^2\theta\right)}{\sin^2\theta} = 1.$$

Therefore the statement of the theorem is true for n=1. Assume that the statement of the theorem is true for the $(n-1)^{st}$ row. Then any two consecutive λ-Farey fractions in the n^{th} row are of the type

$$\frac{\frac{a\sin m\theta+b\sin(m+1)\theta}{\sin\theta}}{\frac{c\sin m\theta+d\sin(m+1)\theta}{\sin\theta}}, \frac{\frac{a\sin(m+1)\theta+b\sin(m+2)\theta}{\sin\theta}}{\frac{c\sin(m+1)\theta+d\sin(m+2)\theta}{\sin\theta}},$$

for some $0 \le m \le q-2$, where $\frac{a}{c}$ and $\frac{b}{d}$, $\frac{b}{d}$ is to the left of $\frac{a}{c}$, are consecutive fractions in the $(n-1)^{st}$ row. If $\frac{a'}{c'} := \frac{\frac{a\sin(m+1)\theta + b\sin(m+2)\theta}{\sin\theta}}{\frac{c\sin(m+1)\theta + d\sin(m+2)\theta}{\sin\theta}}$, and $\frac{b'}{d'} := \frac{\frac{a\sin m\theta + b\sin(m+1)\theta}{\sin\theta}}{\frac{c\sin m\theta + d\sin(m+1)\theta}{\sin\theta}}$, then a simple computation shows that

$$a'd' - b'c' \;=\; (ad-bc)\left(\frac{\sin^2(m+1)\theta - \sin m\theta\sin(m+2)\theta}{\sin^2\theta}\right) = 1.$$

Therefore the statement of the theorem is true for the n^{th} row, and the theorem is proved by mathematical induction.

Example 5.8 *For $q = 3$, $\lambda = 2cos\left(\frac{\pi}{3}\right) = 1$, and $\theta = \frac{\pi}{3}$, the first row of the λ-Farey sequence is*

$$\frac{0}{1}\;\frac{\sin(q-2)\theta/\sin\theta}{\sin(q-1)\theta/\sin\theta} = \frac{\lambda}{1} = \frac{1}{1}.$$

Hence the first row of the λ-Farey sequence is, as expected,

$$\frac{0}{1} \qquad \frac{1}{1}$$

If $\frac{a}{c}$ and $\frac{b}{d}$ are two consecutive λ-Farey fractions in the $(n-1)^{st}$ row, where $\frac{b}{d}$ is to the left of $\frac{a}{c}$, then the λ-Farey fractions in the n^{th} row which are between $\frac{b}{d}$ and $\frac{a}{c}$ are

$$\frac{b}{d},\quad \frac{(a\sin(q-2)\theta + b\sin(q-1)\theta)/\sin\theta}{(c\sin(q-2)\theta + d\sin(q-1)\theta)/\sin\theta},\quad \frac{a}{c}$$

or

$$\frac{b}{d},\quad \frac{a+b}{c+d},\quad \frac{a}{c}.$$

This is in a total agreement with the original definition of the Farey sequence. Hence this justifies the name.

Now we want to investigate the relationship between the λ-Farey sequences and the Hecke groups. Observe that

(i) The Hecke group $H(\lambda)$ is generated by the transformations $P_\lambda := TS_\lambda^{-1}$ and T.

(ii) If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $H(\lambda)$, then either M or MT equals

$$S_\lambda^n \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{or} \quad S_\lambda^n,$$

for some $n \in \mathbb{Z}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $0 \leq \frac{\alpha}{\gamma},\ \frac{\beta}{\delta} \leq \lambda$.

Theorem 5.4 *Let $\lambda = 2cos(\frac{\pi}{q})$, with $q \in \mathbb{N}_{\geq 3}$. Then, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\lambda)$ if and only if for some $k \in \{0, 1\}$, either $MT^k = S_\lambda^n$ for some $n \in \mathbb{Z}$ or there exist two consecutive λ-Farey fractions $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ and an integer n such that*

$$MT^k = S_\lambda^n \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Proof: From the above observations we can assume, without loss of generality, that $0 \leq \frac{a}{c},\ \frac{b}{d} \leq \lambda$. One can prove by mathematical induction that

$$P_\lambda^m = \begin{pmatrix} \frac{\sin(m-1)\theta}{\sin\theta} & -\frac{\sin m\theta}{\sin\theta} \\ \frac{\sin m\theta}{\sin\theta} & -\frac{\sin(m+1)\theta}{\sin\theta} \end{pmatrix} \text{(as a transformation)},$$

for $m = 1, 2, 3, ..., q$. (See (1.43) on p.21).

($\Longleftarrow$) Suppose that $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ are two λ-Farey fractions and n is an integer. We want to show that

$$M = S_\lambda^n \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H(\lambda).$$

It is enough to show that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H(\lambda)$. First let us show the truth of the statement when $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ are in the first row of the λ-Farey sequence. In this case one can easily show that there exists an integer m such that

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = P_\lambda^m \in H(\lambda)$. Now assume that if $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ are in the $(n-1)^{st}$ row of the λ-Farey sequence, then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H(\lambda)$. We want to show that this is true for the n^{th} row. Any two consecutive λ-Farey fractions of the n^{th} row are of the type

$$\frac{\beta}{\delta} = \frac{\frac{a\sin(m-1)\theta + b\sin m\theta}{\sin\theta}}{\frac{c\sin(m-1)\theta + d\sin m\theta}{\sin\theta}}, \quad \frac{\frac{a\sin m\theta + b\sin(m+1)\theta}{\sin\theta}}{\frac{c\sin m\theta + d\sin(m+1)\theta}{\sin\theta}} = \frac{\alpha}{\gamma}$$

for some $m = 1, 2, 3, ..., q-1$ and $\frac{b}{d}$ and $\frac{a}{c}$ are two consecutive λ-Farey fractions of the $(n-1)^{st}$ row. Then by the induction hypothesis

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\lambda).$$

This implies

$$\begin{pmatrix} \frac{a\sin m\theta + b\sin(m+1)\theta}{\sin\theta} & \frac{a\sin(m-1)\theta + b\sin m\theta}{\sin\theta} \\ \frac{c\sin m\theta + d\sin(m+1)\theta}{\sin\theta} & \frac{c\sin(m-1)\theta + d\sin m\theta}{\sin\theta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} P_\lambda^m T.$$

Therefore $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H(\lambda)$.

($\Longrightarrow$) Suppose that $M \in H(\lambda)$. Then by remark we made we may assume that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $0 \leq \frac{a}{c}, \frac{b}{d} \leq \lambda$. Since $M = P_\lambda^{m_1} T P_\lambda^{m_2} T ... T P_\lambda^{m_r}$ for some $m_1, m_2, ..., m_{r-1} \in \mathbb{Z}_{\neq 0}$, $m_1 \neq q-1$ and $m_r \in \mathbb{Z}$, we can easily show that $\frac{a}{c}$ and $\frac{b}{d}$ are two consecutive λ-Farey fractions appearing in the k^{th} row for some $1 \leq k \leq r$. Therefore this completes the theorem.

Remark 5.3 *The above theorem is equivalent to Theorem 1 of [3].*

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