Co-Poisson coalgebras and (co-)Poisson Hopf algebras

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Our Motivation

• Poisson structure (or algebra) is a very active subject of research in mathematics (and mathematical physics) such as: differential geometry, Lie groups, quantum groups, non-commutative geometry, non-commutative algebra and representation theory.

• Co-Poisson structure (or coalgebra) is a dual concept of Poisson structure in categorial point of view.
It arises also in mathematics and mathematical physics naturally.

• The category of connected and simply-connected Lie groups is equivalent to the category of finite-dimensional Lie algebras.

• $\mathcal{O}(G) \cong U(\mathfrak{g})^\circ$, where $\mathfrak{g}$ is the corresponding Lie algebra of Lie group G, $U(\mathfrak{g})^\circ$ is the Hopf dual of the enveloping algebra $U(\mathfrak{g})$.

• A Lie group G is a **Poisson Lie group** if and only if $\mathcal{O}(G)$ is a Poisson Hopf algebra.

• The category of connected and simply-connected Poisson Lie groups is equivalent to the category of finite-dimensional Lie bialgebras.

• The Lie bialgebra structures on any Lie algebra $\mathfrak{g}$ is in one-to-one correspondence with the co-Poisson Hopf structures on $U(\mathfrak{g})$.

• In this case, $\mathcal{O}(G) \cong U(\mathfrak{g})^\circ$ as Poisson Hopf algebra, where $\mathfrak{g}$ is the corresponding Lie bialgebra of Poisson Lie group G.

• To quantize a Lie group or Lie algebra one should equip it with an extra structure, namely, a Poisson Lie group structure or Lie bialgebra structure, respectively.

• Therefore co-Poisson structure naturally appears in the theory of quantum groups and in mathematical physics.

- Let $t_n : V^{\otimes n} \to V^{\otimes n}$ be the map $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$.

- Suppose (A, μ, η) is an algebra. $[-,-] = \mu - \mu \circ t_2$, i.e., $[a, b] = ab - ba$ is the commutator.

- Suppose (C, Δ, ε) is a coalgebra.

$$\Delta(c) = \sum c_1 \otimes c_2 \text{ and } (\Delta \otimes 1)\Delta(c) = \sum c_1 \otimes c_2 \otimes c_3.$$

- Let $\Delta^{(2)} = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : C \to C \otimes C \otimes C$, and $\Delta' = \Delta - t_2 \circ \Delta$ be the cocommutator.

Poisson algebra

Definition 2.1

An algebra A equipped with a linear map $\{-,-\} : A \otimes A \to A$ is called a **Poisson algebra** *if*

1. *A with $\{-,-\} : A \otimes A \to A$ is a Lie algebra;*
2. *$\{a,-\} : A \to A$ is a derivation with respect to the multiplication of A for all $a \in A$, that is, $\{a, bc\} = \{a, b\}c + b\{a, c\}$ for all $b, c \in A$.*

Poisson algebra

Definition 2.2

An algebra (A, μ, η) *equipped with a linear map* $p : A \otimes A \to A$ *is called a* **Poisson algebra** *if*

1. A *with* $p : A \otimes A \to A$ *is a Lie algebra, i.e.,*

$$p \circ (1 + t_2) = 0, \qquad \textit{(skew-symmetric)}$$
$$p \circ (p \otimes 1) \circ (1 + t_3 + t_3^2) = 0; \qquad \textit{(Jacobi identity)}$$

2. $p(1 \otimes \mu) = \mu(p \otimes 1) - \mu(1 \otimes p)t_3^2. \qquad \textit{(Leibnitz rule)}$

Remark 2.3

We don't assume that A *is commutative here.*
The following is a result of Farkas and Letzter [FL, Theorem 1.2].

Proposition 2.4

Suppose that A is a prime Poisson algebra with $\{A, A\} \neq 0$ and $[A, A] \neq 0$. Then, for any $a, b \in A$,

1. *the following map is a bimodule isomorphism* $f_{a,b} : A[a, b]A \to A\{a, b\}A, x[a, b]y \mapsto x\{a, b\}y$.
2. *In the Martindale ring of quotients of A, $\{x, y\} = \alpha[x, y]$ where α is the element represented by any $f_{a,b} \neq 0$.*

It follows that there is no nontrivial Poisson algebra structure on any simple algebras such as $A_n(k)$ and $M_n(k)$.

co-Poisson coalgebra

Definition 2.5

A coalgebra (C, Δ, ε) equipped with a linear map $q : C \to C \otimes C$ is called a **co-Poisson coalgebra** *if*

1. *C with $q : C \to C \otimes C$ is a Lie coalgebra, i.e.,*

$$(1 + t_2) \circ q = 0, \qquad \textit{(skew-symmetric)}$$

$$(1 + t_3 + t_3^2) \circ (q \otimes 1) \circ q = 0; \qquad \textit{(co-Jacobi identity)}$$

2. $(1 \otimes \Delta)q = (q \otimes 1)\Delta - t_3(1 \otimes q)\Delta. \qquad$ *(co-Leibnitz rule)*

The cocommutator Δ' gives a co-Poisson coalgebra structure on any coalgebra (C, Δ, ε).

In a co-Poisson coalgebra (C, q), we use the sigma notation

$$q(c) = \sum c_{(1)} \otimes c_{(2)} \text{ and,}$$

$$(q \otimes 1)q(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$

where $\sum$ is also often omitted in the computations.

Then,

$$(1 \otimes q)q(c) = (1 \otimes q)(-c_{(2)} \otimes c_{(1)}) = -c_{(3)} \otimes c_{(1)} \otimes c_{(2)} = c_{(3)} \otimes c_{(2)} \otimes c_{(1)}.$$

Remark 2.6

By using the sigma notation, the co-Leibnitz rule reads as

$$c_{(1)} \otimes c_{(2)1} \otimes c_{(2)2} = c_{1(1)} \otimes c_{1(2)} \otimes c_2 - c_{2(2)} \otimes c_1 \otimes c_{2(1)},$$

for all $c \in C$.

It is equivalent to

$$c_{(1)1} \otimes c_{(1)2} \otimes c_{(2)} = c_1 \otimes c_{2(1)} \otimes c_{2(2)} - c_{1(2)} \otimes c_2 \otimes c_{1(1)},$$

i.e.,

$$(\Delta \otimes 1)q = (1 \otimes q)\Delta - t_3^2(q \otimes 1)\Delta \qquad \text{(co-Leibnitz rule)}$$

Remark 2.7

If C is cocommutative, then the co-Leibnitz rule is equivalent to

$$(\Delta \otimes 1)q = (1 - t_3)(1 \otimes q)\Delta \qquad \textit{(co-Leibnitz rule)}$$

There is no non-trivial co-Poisson coalgebra structure on any group algebra $k[G]$ (By checking the co-Leibniz rule.)

Dual to $\{a, 1\} = 0$ (because $\{a, -\}$ is a derivation), that is, $p(1 \otimes \eta) = 0$ in a Poisson algebra.

Proposition 2.8

Let $(C, \Delta, \varepsilon, q)$ be a co-Poisson coalgebra. Then $(\varepsilon \otimes 1) \circ q = (1 \otimes \varepsilon) \circ q = 0$, *i.e.,* $\varepsilon(h_{(1)})h_{(2)} = h_{(1)}\varepsilon(h_{(2)}) = 0$.

Poisson Hopf algebra

Definition 2.9

A Hopf algebra $(H, \mu, \eta; \Delta, \varepsilon; S)$ with a linear map $\{-,-\} : H \otimes H \to H$ is called a **Poisson Hopf algebra** *if*

1. $(H, \{-,-\})$ *is a Poisson algebra;*
2. *The structures are compatible: for all $a, b \in H$,*

$$\Delta(\{a, b\}) = \sum \{a_1, b_1\} \otimes a_2 b_2 + \sum a_1 b_1 \otimes \{a_2, b_2\} \quad (2.1)$$

.

Let A and B be two Poisson algebras. An algebra morphism $f : A \to B$ is a **Poisson algebra morphism** if $fp_A = p_B(f \otimes f)$, that is, $f(\{a, b\}_A) = \{f(a), f(b)\}_B$ for all $a, b \in A$.

Remark 2.10

Let A and B be two commutative Poisson algebra. Then there is a Poisson structure on $A \otimes B$ given by

$$\{a \otimes b, a' \otimes b'\} = \{a, a'\} \otimes bb' + aa' \otimes \{b, b'\}$$

for all $a, a' \in A$ and $b, b' \in B$.

If H is commutative, then the cpmpatible condition (2.1) in Definition 2.9 means that $\Delta : H \to H \otimes H$ is a Poisson algebra morphism.

Proposition 2.11 (L. I. Korogodski, Y. S. Soibelman 1998)

Let H be a Poisson Hopf algebra. Then,

1. *the counit $\varepsilon : H \to k$ is a Poisson algebra morphism.*
2. *If H is commutative, then $\Delta : H \to H \otimes H$ is a Poisson algebra morphism.*
3. *If H is commutative, then the antipode $S : H \to H$ is a Poisson algebra anti-morphism.*

Proposition 2.12

Let $\mathfrak{g}$ be a non-abelian Lie algebra over a field of characteristic $\neq 2$. Then there is no nontrivial Poisson Hopf structure on $U(\mathfrak{g})$.

Proposition 2.13

There is no nontrivial Poisson Hopf structure on any group algebra $k(G)$.

co-Poisson Hopf algebra

Definition 2.14

A Hopf algebra $(H, \mu, \eta; \Delta, \varepsilon; S)$ equipped with a linear map $q : H \to H \otimes H$ is called a **co-Poisson Hopf algebra** *if*

1. *H with $q : H \to H \otimes H$ is a co-Poisson coalgebra.*
2. *q is a Δ-derivation, i.e., for all $a, b \in H$,*

$$q(ab) = q(a)\Delta(b) + \Delta(a)q(b) \tag{2.2}$$

.

Let C and D be two co-Poisson coalgebras. A coalgebra morphism $g : C \to D$ is called a **co-Poisson coalgebra morphism** if $(g \otimes g)q_C = q_D g$.

Remark 2.15

Let C and D be two cocommutative co-Poisson coalgebras. Then $C \otimes D$ has a co-Poisson structure $q_{C\otimes D}$ being defined as

$$C \otimes D \xrightarrow{(1\otimes\tau\otimes1)(q_C\otimes\Delta_D+\Delta_C\otimes q_D)} C \otimes D \otimes C \otimes D, \ \textit{that is},$$

$$q_{C\otimes D}(c \otimes d) = c_{(1)} \otimes d_1 \otimes c_{(2)} \otimes d_2 + c_1 \otimes d_{(1)} \otimes c_2 \otimes d_{(2)}.$$

Hence (2.2) in Definition 2.14 is equivalent to say $\mu : H \otimes H \to H$ is a co-Poisson coalgebra morphism.

Proposition 2.16

Let $(H, \mu, \eta; \Delta, \varepsilon; S, q)$ be a co-Poisson Hopf algebra. Then

1. *The unit η is a co-Poisson coalgebra morphism.*
2. *If H is cocommutative, then $\mu : H \otimes H \to H$ is a co-Poisson coalgebra morphism.*
3. *If H is cocommutative, then S is a co-Poisson coalgebra anti-morphism.*

Example 2.17

Let $H_4 = k\langle 1, g, x, gx \mid g^2 = 1, x^2 = 0, xg = -gx\rangle$ *be the 4-dimensional Sweedler's Hopf algebra with* char $k \neq 2$.

1. *Every Poisson algebra structure on* H_4 *is given by* $\{g, x\} = \lambda x + \mu gx$ *for some* $\lambda, \mu \in k$.
2. *There is no nontrivial Poisson Hopf algebra structure on* H_4.
3. *Every co-Poisson structure on* H_4 *is given by* $q(1) = q(g) = 0$, $q(x) = \alpha\Delta'(x)$, $q(gx) = \beta\Delta'(gx)$ *for some* $\alpha, \beta \in k$.
4. *There is no nontrivial co-Poisson Hopf algebra structure on* H_4.

Proposition 3.1

Let C be a coalgebra, $q : C \to C \otimes C$ be a linear map. Then, (C, q) is a co-Poisson coalgebra $\Leftrightarrow$ C^* with $q^* : C^* \otimes C^* \to C^*$ is a Poisson algebra.

If $g : C \to D$ is a co-Poisson coalgebra morphism, then $g^* : D^* \to C^*$ is a Poisson algebra morphism.

Recall for an algebra A,
$A^\circ = \{f \in A^* \mid \ker f \text{ contains a cofinite (left/right) ideal } \mathbf{I} \text{ of } A\}$.

Proposition 3.2

Let A be a Poisson algebra. If A is a left or right noetherian, then A° is a co-Poisson coalgebra.

Example 3.3

Let $A = k[x_1, x_2, \cdots, x_n, \cdots]$ be a polynomial algebra with variables $\{x_i \mid i \geq 1\}$. Let $p(x_i \otimes x_j) = \{x_i, x_j\} = 1$ for all $i < j$. Then p gives a Poisson algebra structure on A. Let $\varepsilon : A \to k, x_i \mapsto 0$ be the augmentation map. Then $\varepsilon \in A^\circ$, but $p^*(\varepsilon) = \varepsilon p \notin A^\circ \otimes A^\circ \cong (A \otimes A)^\circ$.

Theorem 3.4 (L. I. Korogodski, Y. S. Soibelman 1998)

Let H be a left or right noetherian Poisson Hopf algebra. Then H° is a co-Poisson Hopf algebra.

Theorem 3.4 is stated in [KS, Proposition 3.1.5] without noetherian hypothesis. Without this hypothesis, it is not true as showed in the following example.

Example 3.5

Let $A = k[x_1, x_2, \cdots]$ with the Poisson Hopf algebra structure by letting

$$\{x_1, x_i\} = 0 \text{ for all } i \geq 2,$$

for $1 < i < j \in \mathbb{N}$,

$$\{x_i, x_j\} = \begin{cases} x_1, & j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{-,-\}^*(A^\circ) \not\subseteq A^\circ \otimes A^\circ$, thus $\{-,-\}^*$ is not a co-Poisson Hopf structure on A°.

Theorem 3.6 (L. I. Korogodski, Y. S. Soibelman 1998)

Let H be a co-Poisson Hopf algebra. Then the Hopf dual H° is a Poisson Hopf algebra.

Oh and Park prove that the Hopf dual H° of a co-Poisson Hopf algebra H is a Poisson Hopf algebra when H is an almost normalizing extension over k, suggested by the $U(\mathfrak{g})$ case.

This is true in general. A complete proof is given in [LW] and [Oh] in 2015.

Co-Poisson structures on $k[x_1, \cdots, x_d]$

Let $\mathfrak{g} = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_d$ be the d-dimensional abelian Lie algebra. Then $A = U(\mathfrak{g}) = k[x_1, \cdots, x_d]$ is a Hopf algebra. Note that $\mathfrak{g} = P(A)$.
Let $\mathcal{H}(A)$ be the standard k-basis of A which contains all monic monomials.
In the following, for $a \in A$, $\Delta(a) = \sum a_1 \otimes a_2$ is always assumed to be the expression by the standard k-basis of $k[x_1, x_2, \cdots, x_d]$.

Denote $\mathcal{I} = \oplus_{1 \leq i < j \leq d} k(x_i \otimes x_j - x_j \otimes x_i)$.

Lemma 4.1

Let C be a coalgebra, $X \in C \otimes C$. Then

$$X \in \mathcal{I} \Leftrightarrow (1 + t_2)X = 0 \; and \, (\Delta \otimes 1)(X) = (1 - t_3)(1 \otimes X).$$

Lemma 4.2

Let B be a bialgebra. If $X \in B \otimes B$ is skew-symmetric, then so is $X\Delta(x)$ for any $x \in P(B)$.

Theorem 4.3 (Reciprocity law)

Let $q : A \to A \otimes A$ and $l : A \to A \otimes A$ be two linear maps. Then

$$l(a) = (-1)^{|a_2|} q(a_1)\Delta(a_2) \;\; \textit{for all} \;\; a \in A$$
$$\Updownarrow$$
$$q(a) = l(a_1)\Delta(a_2) \;\; \textit{for all} \;\; a \in A.$$

Theorem 4.4

A linear map $q : A \to A \otimes A$ gives a co-Poisson coalgebra structure on A if and only if there is a linear map $I : A \to A \otimes A$ such that,

1. *The image of I is contained in $\mathcal{I}$.*
2. $q(a) = I(a_1)\Delta(a_2)$ *for all $a \in A$.*
3. *The co-Jacobi identity holds for q.*

Theorem 4.5 (Continued)

In this case, we may assume that, for any $a \in \mathcal{H}(A)$,

$$I(a) = \sum_{1 \le i,j \le d} \lambda_a^{ij} x_i \otimes x_j \in \mathcal{I}$$

with $(\lambda_a^{ij})_{d\times d} \in M_d(k)$ skew-symmetric. Then the co-Jacobi identity holds for q if and only if for all $1 \le i < j < k \le d$ and $a \in A$,

$$\sum_{s=1}^{d} \left(\lambda_{a_1}^{sk} \lambda_{x_s a_2}^{ij} + \lambda_{a_1}^{si} \lambda_{x_s a_2}^{jk} + \lambda_{a_1}^{sj} \lambda_{x_s a_2}^{ki} \right) = 0.$$

Proposition 4.6

Let $A = k[x, y]$. Then there is an one-to-one correspondence between the co-Poisson structures q on A and the linear maps $I : A \to \mathcal{I} = k(x \otimes y - y \otimes x)$, given by

$$(I : A \to \mathcal{I} \subseteq A \otimes A) \mapsto (q : A \to A \otimes A, a \mapsto I(a_1)\Delta(a_2)), \text{ and}$$

$$(q : A \to A \otimes A) \mapsto (I : A \to A \otimes A, a \mapsto (-1)^{|a_2|} q(a_1)\Delta(a_2)).$$

This is dual to [CAP, Proposition 1.8] in some sense.

Proposition 4.7

Any Poisson algebra structure on $A = k[x_1, \cdots, x_d]$ is given by $\{x_i, x_j\} = f_{ij}$ where $\{f_{ij}\}_{d\times d}$ is a skew-symmetric matrix over A such that for all $1 \leq i < j < k \leq d$,

$$\sum_{l=1}^{d} \left(f_{lk} \frac{\partial f_{ij}}{\partial x_l} + f_{li} \frac{\partial f_{jk}}{\partial x_l} + f_{lj} \frac{\partial f_{ki}}{\partial x_l} \right) = 0. \tag{4.1}$$

Proposition 4.8

Any Poisson Hopf structure on $A = k[x_1, \cdots, x_d]$ *is given by*

$$\{x_i, x_j\} = \sum_{l=1}^{d} \lambda_{ij}^l x_l \, (1 \leq i, j \leq d),$$

where $\lambda_{ij}^l = -\lambda_{ji}^l$*, subject to the relations, for any* $1 \leq i < j < k \leq d$ *and any* $1 \leq s \leq d$*,*

$$\sum_{l=1}^{n} \left(\lambda_{ij}^l \lambda_{lk}^s + \lambda_{jk}^l \lambda_{li}^s + \lambda_{ki}^l \lambda_{lj}^s \right) = 0.$$

Theorem 4.9

A linear map $q : A \to A \otimes A$ gives a co-Poisson Hopf structure on A if and only if there exists a linear map $l : A \to \mathcal{I}$, such that for any $a \in A$,

1. $l(a) = 0$ *if* $a \neq x_i \, (1 \leq i \leq d)$.
2. $q(a) = l(a_1)\Delta(a_2)$.
3. *the co-Jacobi identity holds.*

Theorem 4.10

Any co-Poisson Hopf structure q on A is given by

$$q(x_s) = \sum_{1 \le i,j \le d} \lambda_s^{ij} x_i \otimes x_j \ (1 \le s \le d),$$

with $\lambda_s^{ij} = -\lambda_s^{ji}$, *subject to the relations, for any* $0 \le i < j < k \le d$ *and* $1 \le s \le d$,

$$\sum_{l=1}^{d} \left(\lambda_s^{lk} \lambda_l^{ij} + \lambda_s^{li} \lambda_l^{jk} + \lambda_s^{lj} \lambda_l^{ki} \right) = 0.$$

Example 4.11

Let $A = k[x, y]$. Then there is an one-to-one correspondence between the co-Poisson Hopf structures q on A and $(l_x, l_y) \in \mathcal{I} \times \mathcal{I}$, given by

$$q \mapsto (q(x), q(y)), \ \ and$$
$$(l_x, l_y) \mapsto q : x^n y^m \mapsto n l_x \Delta(x^{n-1} y^m) + m l_y \Delta(x^n y^{m-1}).$$

Theorem 4.12

Suppose char $k = 0$. *Let* $\tilde{A} = k[[x_1, x_2, \cdots, x_d]]$ *be the algebra of formal power series and* $A = k[x_1, \cdots, x_d]$. *Then there is an one-to-one corresponding between Poisson algebra structures on* $\tilde{A}$ *and co-Poisson coalgebra structures on* A.

Theorem 4.13

There is a one-to-one corresponding between Poisson Hopf structures on A and co-Poisson Hopf structures on A. More precisely, assume

$$\{x_i, x_j\} = \lambda_1^{ij} x_1 + \cdots + \lambda_d^{ij} x_d$$

defines a Poisson Hopf structure on A. Let

$$I(x_s) = \sum_{1 \le i,j \le d} \lambda_s^{ij} x_i \otimes x_j$$

for $1 \le s \le d$ and $I(a) = 0$ for all other $a \in \mathcal{H}(A)$. Then $q(a) = I(a_1)\Delta(a_2)$ defines a co-Poisson Hopf structure on A.

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Thank You!