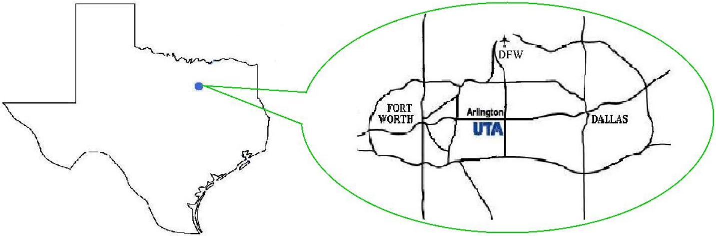
One-Dimensional Line Schemes

Michaela Vancliff

University of Texas at Arlington, USA

http://www.uta.edu/math/vancliff/R   vancliff@uta.edu



Partial support from NSF DMS-1302050.

Motivation

Throughout, $\Bbbk$ = algebraically closed field.

Van den Bergh (early 1990s):
any quadratic algebra on 4 generators with 6 **generic** defining relations has 20 nonisomorphic truncated point modules of length 3 (20 is counted with multiplicity);
if also AS-regular, then it has a 1-parameter family of line modules (in today's language, a 1-dimensional line scheme).

Shelton & Vancliff (late 1990s):
any quadratic algebra on 4 generators with 6 defining relations that

has a **finite** scheme $\mathfrak{z}$ of trunc point modules of length 3 $\Longrightarrow$ **can recover the defining relations from $\mathfrak{z}$**;

or

is AS-reg (+ a few more hyps) & has a **1-diml** line scheme $\mathfrak{L}$ $\Longrightarrow$ **can recover the defining relations from $\mathfrak{L}$**.

Longterm Goal

Classify all quadratic AS-regular algebras A of gldim 4 using $\mathfrak{z}$ or $\mathfrak{L}$.

Subgoal

Identify those $\mathfrak{L}$ of dim $= 1$ where $|\mathfrak{z}| = 20$ (or $|\mathfrak{z}| < \infty$).

Regarding the subgoal: for an embedding in some "appropriate" projective space, can one determine possible degree(s) of $\mathfrak{L}$, at least if $\dim(\mathfrak{L}) = 1$?

I plan to address this question in today's talk.

Notation

Write $\quad A = T(V)/\langle R \rangle \quad$ where

$\dim(V) = 4, \qquad R \subset V \otimes V, \qquad \dim(R) = 6.$

Set-up (Points)

$$\mathbb{P}(V^*) \times \mathbb{P}(V^*) \cong \underbrace{\Omega_1}_{} = \mathbb{P}(\text{rank-1 elements}) \subset \mathbb{P}(V^* \otimes V^*)$$

dim $= 6$
deg $= 20$

$$\mathbb{P}(R^\perp) \subset \mathbb{P}(V^* \otimes V^*)$$

$$\mathfrak{z} \cong \Omega_1 \cap \mathbb{P}(R^\perp)$$

$\dim(\Omega_1 \cap \mathbb{P}(R^\perp)) \geq 6 + 9 - 15 = 0 \implies \mathfrak{z}$ nonempty.

Also, $\deg(\Omega_1 \cap \mathbb{P}(R^\perp)) = (20)(1) = 20$ by Bézout's Thm

and examples of $\mathfrak{z}$ are known where $|\mathfrak{z}| < \infty$.

$\implies |\text{generic } \mathfrak{z}| = 20$ (counted with multiplicity).

So, "generic" R means $\mathbb{P}(R^\perp)$ meets Ω_1 with minimal dimension, in which case $|\mathfrak{z}| = 20$ (counted with multiplicity).

Set-up (Lines)

$$\overbrace{\Omega_2}^{\dim = 11} = \mathbb{P}(\text{rank} \leq 2 \text{ elements}) \subset \mathbb{P}(V \otimes V) \qquad \mathbb{P}^5 \cong \mathbb{P}(R) \subset \mathbb{P}(V \otimes V)$$

AS-regular etc $\Rightarrow$ use Prop 2.8 in T. Levasseur & S.P. Smith's paper

$\Longrightarrow$ line scheme $\mathfrak{L}_R \cong \Omega_2 \cap \mathbb{P}(R) \not\ni$ any rank-1 elements.

$\Longrightarrow$ dim(each irred component of $\mathfrak{L}_R) \geq 11 + 5 - 15 = 1$.

Snag: in order to compare the lines with the points, we wish to view

$\mathfrak{L}_R \subset$ Grassmannian $G(2, V^*)$ = scheme that parametrizes all lines in $\mathbb{P}(V^*)$.

As $G(2, V^*) \overset{\text{Plücker}}{\hookrightarrow} \mathbb{P}(\bigwedge^2 V^*) = \mathbb{P}^5$, we have

Question: what is $\deg(\mathfrak{L}_R)$ viewed in this $\mathbb{P}^5$, at least when $\dim(\mathfrak{L}_R) = 1$?

Joint work with A. Chirvasitu & S.P. Smith $\Rightarrow \deg(\mathfrak{L}_R) = 20$ if $\dim(\mathfrak{L}_R) = 1$.

Approach

Main idea: work with "dual" scheme that has same degree as $\mathfrak{L}_R$ in a "dual" $\mathbb{P}^5$.

i.e., let $\mathfrak{L}_R^\perp =$ scheme in $G(2, V)$ that parametrizes all dim-2 $Q \subset V$ such that $(\underbrace{Q \otimes V}_{\text{rank-2 element in } \mathbb{P}(V \otimes V)}) \cap R \neq 0$.

$$\mathfrak{L}_R \cong \mathfrak{L}_R^\perp \subset G(2, V) \xrightarrow{\text{Plücker}} \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$$

& the map $\mathbb{P}(\bigwedge^2 V) \to \mathbb{P}(\bigwedge^2 V^*)$ is homogeneous of degree 1, so

$$\deg(\mathfrak{L}_R^\perp) = \deg(\mathfrak{L}_R).$$

Main Result

Theorem [Chirvasitu, Smith, V]

Let V be a 4-dimensional vector space, $R \subset V \otimes V$, where $\dim(R) = 6$, & let $\mathfrak{L}_R^\perp$ be the scheme whose reduced variety is

$$\{Q \in G(2, V) : (Q \otimes V) \cap R \neq 0\}.$$

(Note: no hypothesis on $T(V)/\langle R \rangle$ being regular or having good Hilbert series, etc.)

(a) $\dim(\text{each irred component of } \mathfrak{L}_R^\perp) \geq 1$;

(b) $\{\mathfrak{L}_R^\perp : R \subset V \otimes V,\ \dim(R) = 6,\ \dim(\mathfrak{L}_R^\perp) = 1\}$ is a flat family;

(c) if $\text{char}(\Bbbk) \neq 2$ & if $\dim(\mathfrak{L}_R^\perp) = 1$, then $\deg(\mathfrak{L}_R^\perp) = 20$, where $\mathfrak{L}_R^\perp \hookrightarrow \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$.

The lack of homological hypotheses means the theorem is a result about 6-dimensional subspaces of the space of 4×4 matrices.

(And when the algebra is not regular, the schemes $\mathfrak{L}_R$ and $\mathfrak{L}_R^\perp$ parametrize the truncated right line modules of dimension three.)

Idea of Proof

- Prove (a) and (b), and then use (b) (i.e., flatness) to prove $\deg(\mathfrak{L}_R^{\perp})$ is a constant for all R that satisfy the hypotheses of the theorem where $\dim(\mathfrak{L}_R^{\perp}) = 1$;
- then exhibit an example that has $\deg(\mathfrak{L}_R^{\perp}) = 20$.

For the example, we used an algebra I had previously studied with R. Chandler, but that work assumed $\text{char}(\Bbbk) = 0$. So we computed the degree assuming $\text{char}(\Bbbk) \neq 2$.

Will now present 3 examples of $\mathfrak{L}_R$, where $T(V)/\langle R \rangle$ is regular & $\text{char}(\Bbbk) = 0$.

1st Example [Chandler, V]

Let $\gamma,\ i \in \Bbbk^\times,\ \ i^2=-1,\ \ V=$ span of $x_1,\ldots,x_4$, & $R=$ span of:

$$\begin{array}{lll} x_4x_1 - ix_1x_4, & x_3^2 - x_1^2, & x_3x_1 - x_1x_3 + x_2^2, \\ x_3x_2 - ix_2x_3, & x_4^2 - x_2^2, & x_4x_2 - x_2x_4 + \gamma x_1^2. \end{array}$$

If $\gamma(\gamma^2-4)\neq 0$, then $|\mathfrak{z}|=20$ and can be grouped naturally in $\mathbb{P}^3$ into 2 sets of 2 points and 4 sets of 4 points (call latter type generic points).

If $\gamma(\gamma^2-16)\neq 0$, then $\mathfrak{L}_R=$ union of 6 subschemes in $\mathbb{P}^5$:

- 1 nonplanar deg-4 elliptic curve in a $\mathbb{P}^3$ (i.e., spatial elliptic curve),
- 4 planar elliptic curves,
- a subscheme in a $\mathbb{P}^3$ consisting of the union of 2 nonsingular conics.

Each of the 16 generic points of $\mathfrak{z}$ lies on 6 distinct lines parametrized by $\mathfrak{L}_R$ (1 line from each of the above 6 subschemes).

Each of the remaining 4 points lies on infinitely many lines parametrized by $\mathfrak{L}_R$.

2nd Example [Derek Tomlin, V]

Let $\alpha \in \mathbb{k}$, $\alpha(\alpha^2 - 1) \neq 0$, $V =$ span of $x_1, \ldots, x_4$, $R =$ span of:

$$x_1x_3 + x_3x_1, \qquad x_2x_3 - x_3x_2, \qquad x_2x_4 + x_4x_2 - x_3^2,$$
$$x_1x_4 + x_4x_1, \qquad x_2^2 - x_4^2, \qquad 2x_2^2 + \alpha x_3^2 - x_1^2.$$

$|\mathfrak{z}| = 20$ & the points can be grouped naturally in $\mathbb{P}^3$ into 10 sets of 2 points.

$\mathfrak{L}_R =$ union of 6 subschemes in $\mathbb{P}^5$:

- 1 nonplanar deg-4 elliptic curve in a $\mathbb{P}^3$, (i.e., spatial elliptic curve),
- 1 nonplanar deg-4 rational curve (with 1 singular point) in a $\mathbb{P}^3$,
- 2 planar elliptic curves,
- 2 subschemes, each of which consists of the union of a nonsingular conic and a line (that meets the conic in 2 distinct points).

$\exists$ 16 points $p \in \mathfrak{z}$ such that p lies on 6 lines (ctd with mult) of those parametrized by $\mathfrak{L}_R$ (1 line from each of the above 6 subschemes, but some lines belong to more than 1 family).
Each of the remaining 4 points lies on infinitely many lines parametrized by $\mathfrak{L}_R$.

3rd Example [Chirvasitu, Smith, Derek Tomlin]

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{k} \setminus \{0, -1, 1\}$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$,
$V =$ span of $x_1, \ldots, x_4$, $R =$ span of:

$$x_4x_i - x_ix_4 - \alpha_i(x_jx_k - x_kx_j), \qquad x_4x_i + x_ix_4 - \alpha_i(x_jx_k + x_kx_j),$$

where (i, j, k) cycles through $(1, 2, 3)$.

$|\mathfrak{z}| = 20$ & the points can be grouped naturally in $\mathbb{P}^3$ into 6 sets of 2 points and 8 other points.

$\mathfrak{L}_R =$ union of 7 irreducible subschemes in $\mathbb{P}^5$:

- 3 nonplanar deg-4 elliptic curves in a $\mathbb{P}^3$, (i.e., spatial elliptic curves E_1, E_2, E_3),
- 4 nonsingular conics.

$\exists$ 16 points $p \in \mathfrak{z}$ such that $p \in 3$ lines, each given by a distinct conic, and $p \in 3$ other lines, each given by a distinct elliptic curve.
The remaining 4 points $p \in \mathfrak{z}$ are such that $p \in 2$ lines given by E_i, $i = 1, 2, 3$ (6 total).

1st 2 examples are graded skew Clifford algebras, but 3rd does not appear to be related to a graded skew Clifford algebra.

Conjectures

Lines Conjecture

One of the generic classes of quadratic AS-regular algebra of gldim 4 has line scheme that consists of

- the union of 2 deg-4 spatial elliptic curves & 4 planar elliptic curves ($(2)(4) + (4)(3) = 20$);

and another generic class has line scheme that consists of

- the union of 4 deg-4 spatial elliptic curves & 2 nonsingular conics ($(4)(4) + (2)(2) = 20$).

Points Conjecture

Generic $R \implies a \otimes b \in R^{\perp}$ iff $b \otimes a \in R^{\perp}$.

Future Work?

Identify more $\mathfrak{L}_R$, perhaps where $|\mathfrak{z}| < 20$?

Can $\mathfrak{L}_R =$ a union of lines? 20 distinct lines? (even if A not AS-regular)

- M. Artin, J. Tate & M. Van den Bergh, Some Algebras Associated to Automorphisms of Elliptic Curves, *The Grothendieck Festschrift* **1**, 33-85, Eds. P. Cartier et al., Birkhäuser (Boston, 1990).
- R. G. Chandler & M. Vancliff, The One-Dimensional Line Scheme of a Certain Family of Quantum $\mathbb{P}^3$s, *J. Algebra* **439** (2015), 316-333.
- A. Chirvasitu & S. P. Smith, Exotic Elliptic Algebras of Dimension 4 (with an Appendix by Derek Tomlin), *Adv. Math.* **309** (2017), 558-623.
- A. Chirvasitu, S. P. Smith & M. Vancliff, A Geometric Invariant of 6-Dimensional Subspaces of 4×4 Matrices, 1st draft at arXiv:1512.03954.
- D. Tomlin & M. Vancliff, The One-Dimensional Line Scheme of a Family of Quadratic Quantum $\mathbb{P}^3$s, preprint 2017. (arXiv:1705.10426)
- M. Vancliff, The Interplay of Algebra and Geometry in the Setting of Regular Algebras, *in* "Commutative Algebra and Noncommutative Algebraic Geometry," *MSRI Publications* **67** (2015), 371-390.

Appendix: $G(2, V) \xhookrightarrow{\text{Plücker}} \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$

Write $V = \bigoplus_{i=1}^4 \Bbbk x_i, \quad u = \sum_{i=1}^4 u_i x_i, \; w = \sum_{i=1}^4 w_i x_i \in V,$

$$Q = \Bbbk u \oplus \Bbbk w \mapsto u \wedge w = \sum_{i<j} N_{ij}\, x_i \wedge x_j \in \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$$

& the N_{ij} are the 2×2 minors of $\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}$.

The 6 N_{ij} are homogeneous coordinates on $\mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$.