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MEASURE OF PLANES SEPARATING CONVEX BODIES IN THREE DIMENSIONS

A Dissertation
Submitted to
the Temple University Graduate Board

---

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
B. Clark Loveridge
January, 2002

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ABSTRACT

MEASURE OF PLANES SEPARATING CONVEX BODIES IN THREE DIMENSIONS

B. Clark Loveridge

DOCTOR OF PHILOSOPHY

Temple University, January, 2002

Professor Eric Grinberg, Chair

Here we primarily build on the work of R.V. Ambartzumian in the field of combinatorial integral geometry who in ground-breaking work in the 1970s developed formulas for computing the measure of planes partitioning a finite number of points in three-dimensional space. We also secondarily make slight additions to the work of Ambartzumian and others on computing the measure of lines partitioning a finite number of points in the plane.

For the three-dimensional case some loose definitions are helpful. We make use of Ambartzumian's wedge function. A wedge function on the edge of a polyhedron is half the length of the edge times the angle between the adjoining faces. A support plane for a convex body intersects the boundary but not the interior of the convex body. Given two disjoint closed convex bodies we take

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the envelope to be the portion of the envelope of separating double support planes bounded by the points of separating double support. We take the caps to be the nearby portions of the original surfaces bounded by the points of separating double support.

We show here that the measure of planes separating two disjoint convex polyhedra in three-dimensional space is equal to the sum of the wedge functions over the envelope minus the sum of the wedge functions over the caps. Analogously we also show that the measure of planes separating at least certain pairs of smooth convex bodies in three-dimensional space is equal to the total absolute mean curvature over the envelope minus the total absolute mean curvature over the caps.

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I thank friends and family members for bearing with me these last five years while I worked on my dissertation.

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To my parents

Ben and Margaret Loveridge

for their unwavering devotion,

to my first calculus teacher

Joseph McCully

who made calculus fun,

and to my girlfriend Lee Friedman

who dated me even though

I was working on my dissertation,

and made my life fun.

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CHAPTER 1

INTRODUCTION AND HISTORICAL BACKGROUND

1.1 Overview

In this paper we develop some formulas for computing measures on lines and planes with an emphasis on computing the measure of planes separating two convex bodies in R^3. Although formulas already exist for computing measures of planes we seek to express the measure in terms of geometric invariants on the bounding surfaces. For example we seek to prove that the measure of planes separating two smooth convex bodies in R^3 is equal to the integral of absolute mean curvature over the envelope of separating double tangent planes minus the integral of absolute mean curvature over the portions of the original

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surfaces bounded by the envelope. This branch of mathematics was dubbed *combinatorial integral geometry* by its main recent developer R.V. Ambartzumian to whom we are deeply indebted.

1.2 The Buffon Needle Problem

Although our work builds most directly on the work of R.V. Ambartzumian from the 1970s, our work has its roots in a problem solved by Buffon in 1776 (Buffon [1776] 1977). Buffon showed that if one randomly throws a needle on a parquet floor. the probability that the needle hits the crack is

$$\frac{2|\nu|}{\pi d}$$

where $|\nu|$ is the length of the needle and d is the distance between the cracks.

Buffon regarded the cracks on the parquet floor as fixed and the placement of the needle as random but one could equivalently regard the placement of the needle as fixed and the placement of a grid of parallel lines as random. Thus the reformulated problem is the probability that a random line hits the needle which is a measure on the set of lines in the plane.

A desirable property of measures is that they be invariant under rigid motions. It turns out that Buffon's probability regarded as a measure on lines in the plane has this property. It has been shown in fact that any motion invariant measure on lines in the plane is a constant multiple of Buffon's measure.

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The essential factor is the length of the needle. For convenience we take the measure of lines intersecting the needle to be twice the length of the needle. See for example Ambartzumian (1990, 47-50, 107, 123-124).

1.3 Barbier, Crofton, and Sylvester

The next wave of development of combinatorial integral geometry came in the late 1800s. In 1860 Barbier showed that the measure of lines intersecting a compact convex set is equal to the perimeter of the set (Barbier 1860). In 1868 Crofton showed that the measure of lines separating two compact convex sets is equal to the length of a taut crossed string enclosing the two sets minus the length of the perimeters of the sets (Crofton 1868). The taut crossed string has a linear part which may be referred to as the envelope of separating double support lines. See Figure 1.1. In 1890 Sylvester showed that the measures of lines intersecting either n of n compact convex sets or at least 1 of n compact convex sets are integer combinations of lengths of taut strings around the sets. Sylvester did not give a specific formula (Sylvester [1890] 1973).

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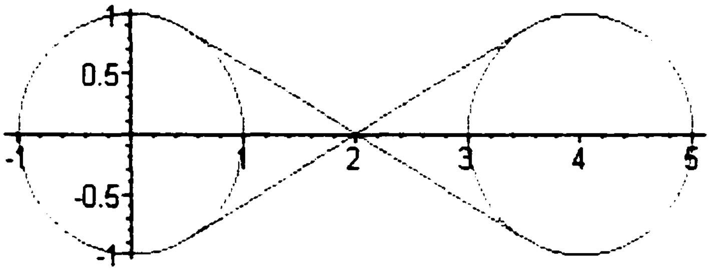

Figure 1.1: Circles and the Envelope

1.4 Minkowski and Ambartzumian

In the 1970s R.V. Ambartzumian in ground-breaking work gave specific formulas for computing measures of lines hitting various combinations of compact convex sets in R^2. Building on prior work by Minkowski and others he also extended some of the results to measures on planes intersecting convex sets in R^3 and to higher dimensions (Ambartzumian 1990, 100-126).

In moving from R^2 to R^3 length is replaced by other geometric invariants. For example the measure of planes intersecting a smooth compact convex body in R^3 is equal to the integral of absolute mean curvature over the boundary. For polyhedral convex bodies in R^3 the mean curvature is replaced by what Ambartzumian referred to as the wedge function. The wedge function on an edge is equal to half the length of the edge times the outer angle of the

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adjoining faces. The measure of planes intersecting a polyhedral compact convex body in R^3 is equal to the total wedge function over the edges.

Both of these results in R^3 were referred to by Ambartzumian (1990, 114, 122) as classical formulas. The result on the measure of planes intersecting a smooth convex body in R^3 was attributed to H. Minkowski by R. Deltheil (1926, 95).

The main focus of this paper is to extend the results of Minkowski and Ambartzumian by expressing the measure of planes separating two convex bodies in R^3 in terms of mean curvature and wedge functions. We will start with some basic definitions in the next sections and then give a more detailed summary of Ambartzumian's results in the following two chapters.

1.5 Motion Invariant Measures

The following theorems are classical results in integral geometry and will be a starting point for some of our subsequent results.

Theorem 1.5.1 *There is a locally finite measure on the set of lines in the plane which is invariant under rigid motions and with the property that the set of lines through a point has measure zero. This measure is unique up to multiplication by a constant factor.*

Proof. See for example Ambartzumian (1990, 47-50).

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Theorem 1.5.2 *There is a locally finite measure on the set of planes in* R^3 *which is invariant under rigid motions and with the property that the set of planes through a point has measure zero. This measure is unique up to multiplication by a constant factor.*

Proof. See for example Ambartzumian (1990, 53-55).

Definition 1.5.3 **Almost all** *planes or* **almost every** *plane means all planes except for a set of planes of measure zero. Almost all lines or almost every line means all lines except for a set of lines of measure zero.*

Theorem 1.5.4 *Let* (x, y, z) *denote a point in* R^3. *Let* $0 \leq \phi \leq \pi/2$. *Let* $0 \leq \theta < 2\pi$. *Let* $w \in R$. *If almost all planes in* R^3 *are parametrized by angle* ϕ *of the upward normal vector with the z-axis, angle* θ *of the angle with the x-axis of the projection of the upward normal vector onto the xy-plane, and point of intersection* w *with the z-axis then the motion invariant measure on the set of planes is*

$$\cos\phi \sin\phi \; d\phi \; dw \; d\theta.$$

For a proof see for example Ambartzumian (1990, 53).

Theorem 1.5.5 *Let* (x, y, z) *denote a point in* R^3. *Let* $0 \leq \phi \leq \pi/2$. *Let* $0 \leq \theta < 2\pi$. *Let* $\rho \in R$. *If almost all planes in* R^3 *are parametrized by angle* ϕ *of the upward normal vector with the z-axis, angle* θ *of the angle with the*

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x-axis of the projection of the upward normal vector onto the xy-plane, and signed distance p from the origin then the motion invariant measure on the set of planes is

$$\sin\phi \, d\phi \, dp \, d\theta.$$

For a proof see for example Ambartzumian (1990, 53).

1.6 The Main Conjecture

Definition 1.6.1 *A* **convex set** *in R^n is a set which contains all points on line segments whose endpoints are in the set. A* **convex body** *in R^n is a convex set with nonempty interior. A map is C^r if it has continuous partial derivatives of up to order r inclusive. The boundary of a convex body in R^n is* **smooth** *if it is C^2. A convex body in R^n is* **strictly convex** *if there is a one-to-one correspondence between outward normal directions and points on the boundary.*

Definition 1.6.2 *A* **support** *plane for a convex body A (with interior) in R^3 is a plane which intersects the boundary but not the interior of A. If A is smooth then the set of support planes is exactly the set of tangent planes of A.*

Definition 1.6.3 *Let A and B be two disjoint closed convex bodies in R^3. An* **envelope** *of separating double support planes for A and B is a compact*

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connected surface with boundary which intersects A and B at points of separating double support, is bounded by the points of separating double support, and whose support planes are exactly the separating double support planes of A and B.

Definition 1.6.4 *Let A and B be two disjoint closed convex bodies in R^3. The* **cap** *of A is the compact connected portion of the boundary of A which includes a closest point to B and which is bounded by the points of separating double support. The cap of B is defined analogously.*

Definition 1.6.5 *Let A be a set in R^3. If A has an edge then a* **wedge function** *on an edge of A is defined to be half the length of the edge times the angle of the adjoining faces.*

Conjecture 1.6.6 *Let A and B be two disjoint compact convex bodies in R^3. Then the measure of planes separating A and B is equal to the total absolute mean curvature/wedge function over the envelope minus the total absolute mean curvature/wedge function over the caps.*

The condition of compactness may be replaced by a weaker condition assuring the existence of the envelope. In what follows we do not prove this conjecture in its most general form but we prove that it is true for polyhedra, pairs of surfaces whose envelope is a cone, and some other classes of surface pairs whose envelope is not necessarily a cone.

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We note that for two convex bodies A and B in R^2 **envelopes** and **caps** may be defined as above but replacing planes with lines. Thus in terms of envelopes and caps the measure of lines separating two convex bodies in R^2 is the length of the envelope minus the length of the caps. Thus the theorem of Crofton referred to in Section 1.3 above may be extended to pairs of noncompact convex bodies A and B in R^2 provided the envelope exists.

1.7 Differential Geometry, Convexity, and Calculus

Although this paper builds primarily on the work of R.V. Ambartzumian in the field of combinatorial integral geometry, we use some notions from the fields of differential geometry, convexity, and calculus. For differential geometry we found Do Carmo (1976) and Gray (1993) to be helpful. For basic notions in convexity we found Thompson (1996, 1-11, 45-52) and Gardner (1995, 1-24)to be particularly helpful.

We also use a version of the implicit function theorem found on page 121 of Abraham, Marsden, and Ratiu (1988) but specialized for our purposes to two dimensions.

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Theorem 1.7.1 *Let $f(\theta, \phi) : R^2 \to R$ be C^r for $r \geq 1$. Assume the partial derivative f_ϕ is non-vanishing at (θ_0, ϕ_0). Let $d = f(\theta_0, \phi_0)$. Then there is a neighborhood U of θ_0 and a unique C^r map $\psi(\theta) : U \to R$ such that $f(\theta, \psi(\theta)) = d$ for all $\theta \in U$.*

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CHAPTER 2

MEASURE OF LINES SEPARATING POINTS IN R^2

2.1 Results of Ambartzumian and Others on Lines in the Plane

Our starting point is some formulas developed by R.V.Ambartzumian for computing the measure of lines partitioning a finite number of points in R^2. Some of Ambartzumian's results in R^2 are summarized below and extended slightly.

Definition 2.1.1 (Ambartzumian 1990, 100) *Let* $P = \{P_i\}_1^n$ *denote a finite set of points in* R^2 *such that no 3 of the points are collinear and* $n \geq 2$*. Almost*

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every line in R^2 partitions P into two disjoint sets according to which half-plane the points lie in. The set of lines in R^2 which form the same partition of P is called an **atom**. *If neither set of the partition of P is empty the atom is called a* **bounded atom**. *The set of unions of bounded atoms forms a* **Radon ring** $r(P)$. *Line segments $\overline{P_i, P_j}$ where $P_i, P_j \in P$ are sometimes referred to as* **needles** *for historical reasons. See the description of the Buffon needle problem in Section 1.2.*

Remark 2.1.2 *Each line in R^2 determines two half-planes. Two points $P_i, P_j \in R^2$ may be used to assign a sign to almost all half-planes in a continuous way as follows. Put a number scale on the line L through P_i and P_j. Assign a $+$ sign to half-planes which contain points of L with arbitrarily large values. Assign a $-$ sign to half-planes which contain points of L with arbitrarily small values.*

Notation 2.1.3 (Ambartzumian 1990, 100-101) *Let $P = \{P_i\}_1^n$ denote a finite set of points in R^2 such that no 3 of the points are collinear and $n \geq 2$. Each line segment $\overline{P_i, P_j}$ where $P_i, P_j \in P$ may be used to assign an orientation to lines in neighboring atoms as follows. First arbitrarily assign a $+$ sign to one of the half-planes determined by the line through $\overline{P_i, P_j}$ and a $-$ sign to the other half-plane. Next infinitesimally displace the line. The displaced line will be in one of four atoms corresponding to the four different ways of*

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assigning +/− signs to P_i *and* P_j. *Denote the displacements of* P_i *and* P_j *as* P_i' *and* P_j' *respectively. If* P_i' *is in the positive half-plane determined by the pair* P_i, P_j *then designate the half-plane containing* P_i *determined by the displaced line with a positive sign. Otherwise designate the half-plane containing* P_i *determined by the displaced line with a negative sign. Extend the orientation continuously to the other lines of the atom. Do this for all four bounding atoms. Let* $(\overset{+}{i}.\overset{+}{j})$ *denote the bounding atom whose positive half-planes contain both points* P_i *and* P_j *and whose lines otherwise form the same partition of* P *as the line through* $\overline{P_i.P_j}$. *Likewise let* $(\overset{-}{i}.\overset{+}{j})$, $(\overset{+}{i}.\overset{-}{j})$, and $(\overset{-}{i}.\overset{-}{j})$ *respectively denote atoms whose designated half-planes contain points* P_i *and* P_j.

Theorem 2.1.4 (Ambartzumian 1990, 100-107) *Let* $P = \{P_i\}_1^n$ *denote a finite set of points in* R^2 *such that no 3 of the points are collinear and* $n \geq 2$. *For each pair* $P_i, P_j \in P$ *let* $(\overset{+}{i}.\overset{+}{j})$, $(\overset{-}{i}.\overset{+}{j})$, $(\overset{+}{i}.\overset{-}{j})$, and $(\overset{-}{i}.\overset{-}{j})$ *respectively denote atoms whose designated half-planes contain points* P_i *and* P_j. *Let* Q *be an element of the Radon ring of* P. *Let* $I_Q(A)$ *be an indicator function on atoms which takes on the value 1 if* $A \subset Q$ *and 0 otherwise. Let*

$$c_{ij}(Q) = I_Q(\overset{+}{i}.\overset{-}{j}) + I_Q(\overset{-}{i}.\overset{+}{j}) - I_Q(\overset{+}{i}.\overset{+}{j}) - I_Q(\overset{-}{i}.\overset{-}{j}).$$

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Let ρ_{ij} be the length of the line segment $\overline{P_iP_j}$. Let $m(Q)$ denote the motion invariant measure on the set of lines in R^2. Then

$$m(Q) = \sum_{i<j} c_{ij}(Q)\rho_{ij}.$$

2.2 A Slight Generalization of Ambartzumian's Results on Lines in the Plane

In order to compute the measure of lines separating polygons it is useful to have a slightly more general formula that allows for the possibility that three or more points will be collinear.

Definition 2.2.1 *Let $P = \{P_i\}_1^n$ denote a finite set of points in R^2 where $n \geq 2$ as above but now we allow for the possibility that three or more points are collinear. Let $P_i, P_j \in P$. Then the pair of points P_i, P_j is called an* **allowable pair** *if no point of P lies on the interior of the line segment $\overline{P_iP_j}$. There are up to four* **allowable bounding atoms** *associated with each allowable pair P_i, P_j as follows. The two atoms which separate P_i and P_j and which otherwise induce the same partition of P as the line through P_i and P_j are allowable. Also the two atoms for which all points collinear with P_i and P_j are in the same half-plane and which otherwise induce the same partition of P as the line through P_i and P_j are allowable.*

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Theorem 2.2.2 *Let $P = \{P_i\}_1^n$ denote a finite set of points in R^2 for $n \geq 2$. For each allowable pair $P_i, P_j \in P$ let $(\overset{+}{i}, \overset{+}{j})$, $(\overset{-}{i}, \overset{+}{j})$, $(\overset{+}{i}, \overset{-}{j})$,* and $(\overset{-}{i}, \overset{-}{j})$ *respectively denote allowable atoms whose designated half-planes contain points P_i and P_j. Let Q be an element of the Radon ring of P. Let $I_Q(A)$ be an indicator function on atoms which takes on the value 1 if $A \subset Q$ and 0 otherwise. Let*

$$c_{ij}(Q) = I_Q(\overset{+}{i}, \overset{-}{j}) + I_Q(\overset{-}{i}, \overset{+}{j}) - I_Q(\overset{+}{i}, \overset{+}{j}) - I_Q(\overset{-}{i}, \overset{-}{j}).$$

Let ρ_{ij} be the length of the line segment $\overline{P_iP_j}$. Let $m(Q)$ denote the motion invariant measure on the set of lines in R^2. Then

$$m(Q) = \sum_{i<j} c_{ij}(Q)\rho_{ij}.$$

Proof. We will use induction on the number of points. If the set P contains only two points then no three points are collinear and the theorem is true by Theorem 2.1.4. Now suppose that the theorem is true if the set P contains k points. We want to prove that it is true if the set P contains $k + 1$ points.

Case I. No three points are collinear. Then the theorem is true by Theorem 2.1.4.

Case II. P contains a set of at least three collinear points. Without loss of generality, order three of the points of the collinear set P_1, P_2, and P_3, assume

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that no points of P are interior to the line segment $\overline{P_1P_2}$, and assume that P_3 is not an interior point to any line segment formed by points of P and collinear with $\overline{P_1P_2}$.

We consider first an atom a_1 that separates two points of a set of three or more collinear points. Without loss of generality assume that the atom a_1 separates P_1 and P_2. Extend the atom a_1 to an atom a_1' of $P\backslash P_3$ by including in the new atom lines that either do or do not separate P_1 and P_3. But any line that separates P_1 and P_2 will also separate P_1 and P_3. Thus the original atom a_1 of P contains exactly the same set of lines as the new atom a_1' of $P\backslash P_3$.

By supposition the theorem is true on $P\backslash P_3$. Thus it will suffice to show that the coefficient of any line segment which has P_3 as an endpoint is zero. But any allowable bounding atom for P_3 would contain lines which intersect the line through $\overline{P_1P_2}$ arbitrarily close to P_3. Such an atom would not separate P_1 and P_2. Thus the indicator function for such a bounding atom at a_1 would be zero. Thus the coefficient of the line segment would be zero.

We next consider an atom a_2 that does not separate any points of a set of three or more collinear points. Label the points P_1, P_2, and P_3 as above. Then a_2 is also an atom of $P\backslash P_2$. By supposition the theorem is true on $P\backslash P_2$. Two of the bounding atoms in $P\backslash P_2$ of $\overline{P_1, P_3}$ will intersect the line segment and thus have coefficient zero. The other two will be bounding atoms for $\overline{P_1, P_2}$

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and $\overline{P_2, P_3}$ in P and thus have the same coefficient in either ring. Thus the coefficient of a_2 will be the same in either ring.

Thus the formula gives the correct measure on atoms of P. Therefore the formula gives the correct measure on an set in the Radon ring of P where P contains $k+1$ points. Therefore by induction on the number of points, the formula is true for any finite number of points. □

2.3 Examples

Example 2.3.1 *Let $\{D_i\}_1^n$ be a set of bounded convex polygonal domains with disjoint closures in R^2 with the property that no three vertices are collinear. The measure of the set K of lines intersecting exactly k of the polygons for $1 \leq k \leq n$ is*

$$\begin{aligned} m(K) &= \sum_i \left(-I_k(\nu_i) + I_{k-1}(\nu_i)\right) |\nu_i| \\ &+ \sum_i \left(I_k(d_i) - 2I_{k-1}(d_i) + I_{k-2}(d_i)\right) |d_i| \\ &+ \sum_i \left(-I_k(s_i) + 2I_{k-1}(s_i) - I_{k-2}(s_i)\right) |s_i| \end{aligned}$$

where I_k or $I_k(\cdot)$ denotes the indicator function on the set of line segments whose continuation intersects the interiors of exactly k of the polygonal domains. ν_i represent the original edges of the polygons. d_i represent line segments connecting vertices of two different polygons and whose continuation

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separates the two polygons, and s_i represent line segments connecting vertices of two different polygons and whose continuation is the boundary of a half-plane containing both of the polygons.

Proof. We will determine coefficients of I_k on a line segment $\overline{P_iP_j}$ by setting them equal to $c_{ij}(K)$ on the four bounding atoms using the four indicator formula. Note that if a line through a line segment intersects the interior of a polygon a small perturbation of the line will still intersect the interior of that polygon by properties of open sets. Likewise if a line through a line segment does not intersect the closure of a polygon then a small enough perturbation of the line will not intersect the closure of a polygon. Thus all four bounding atoms will intersect the interiors of exactly the same polygons as the line through the line segment except perhaps those polygons with a vertex that coincides with an endpoint of the line segment and whose interiors are disjoint from the line through the line segment.

First consider the coefficient of interior line segment of a polygon. By the remark above all four of the bounding atoms intersect exactly the same set of polyhedra. Thus the four indicator functions are either all ones or all zeros. Thus the coefficient of I_k on a wedge of this type is either $1 + 1 - 1 - 1$ or $0 + 0 - 0 - 0 = 0$ for all k.

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Next consider the coefficient of an edge of a polygon. The bounding atom that sends the vertices of the edge to the same half-space as the rest of the vertices of the polygon will not intersect the interior of the polygon. The other three bounding atoms will separate at least one of the vertices of the polygon from the rest of the polygon. Thus the coefficient of I_k will be $0+0-0-1=-1$ and the coefficient of I_{k-1} will be $1+1-1-0=1$ on line segments of this type.

Now consider the coefficients of line segments whose endpoints are on two different polygons and whose continuation separates those two polygons. We will refer to line segments of this type as **separating line segments**. Note that the bounding atom which sends each endpoint of the line segment to the half-plane with the other vertices from the same polygon intersects neither of the two polygons. Note also that the bounding atom which sends each endpoint of the line segment to the opposite half-plane intersects the interior of both of the polygons. Finally there are two bounding atoms which send both endpoints to the same half-plane. Such bounding atoms intersect the interior of exactly one of these two polygons. Thus the coefficient of I_k will be $1+0-0-0=1$. the coefficient of I_{k-2} will be $0+1-0-0=1$ and the coefficient of I_{k-1} will be $0+0-1-1=-2$ on line segments of this type.

Finally consider line segments with endpoints on two polygons which send both of these polygons to the same half-plane. We will refer to these line

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segments as **linking line segments**. There is one bounding atom that sends both endpoints of the line segment to the half-plane which contains the other vertices of the two polygons. Such an atom will intersect neither polygon. Thus the coefficient of I_k on line segments of this type is $0+0-1-0=-1$. There is one bounding atom that will send both endpoints of the line segment to the opposite half-plane. Such an atom will intersect the interiors of both polygons. The other two bounding atoms send exactly one of the endpoints of the line segment to the same half-plane as the other vertices of the two polygons. Such a bounding atom will intersect the interiors of exactly one of these two polygons. Thus the coefficient of I_k will be $0+0-1-0=-1$, the coefficient of I_{k-2} will be $0+0-0-1=-1$, and the coefficient of I_{k-1} will be $1+1-0-0=2$ on line segments of this type.

Thus the types of line segments that will have nonzero coefficients are edges, separating line segments, and linking line segments. Thus for $1 \leq k \leq n$ the measure of the lines hitting exactly k of the n polygons is

$$\sum_i \left(-I_k(\nu_i) + I_{k-1}(\nu_i)\right)|\nu_i|$$
$$+\sum_i \left(I_k(d_i) - 2I_{k-1}(d_i) + I_{k-2}(d_i)\right)|d_i|$$
$$+\sum_i \left(-I_k(s_i) + 2I_{k-1}(s_i) - I_{k-2}(s_i)\right)|s_i|$$

where $\{\nu_i\}$ is the set of line segments on edges, $\{d_i\}$ is the set of separating line segments, and $\{s_i\}$ is the set of linking line segments. □

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Example 2.3.2 *We can extend the formula of Example 2.3.1 to the measure of lines which intersect the convex hull of the set of polygons but none ($k = 0$) of the polygons if we include only line segments which intersect the interior of the convex hull in the summations.*

Example 2.3.3 *If $k = n$ the formula of Example 2.3.1 specializes to*

$$\sum_i I_{n-1}(\nu_i)\,|\nu_i| + \sum_i I_{n-2}(d_i)\,|d_i| + \sum_i I_{n-2}(s_i)\,|s_i|\,.$$

This represents the measure of lines hitting all n of the polygons and agrees with Ambartzumian's example (1990, 108-111) *and is the solution to a problem posed by Sylvester* (1890).

Example 2.3.4 *The measure of lines hitting at least one of the polygons may be obtained by summing the formula of Example 2.3.1 over k for $k = 1 \ldots n$. Much cancellation occurs and the result is*

$$\sum_i I_o(\nu_i)\,|\nu_i| - \sum_i I_o(d_i)\,|d_i| + \sum_i I_o(s_i)\,|s_i|\,.$$

This agrees with Ambartzumian's example (1990, 108-111) *and is the solution to a problem posed by Sylvester* (1890).

Example 2.3.5 *The examples above may be extended to replace some or all of the polygonal domains with line segments with slight modification. Specifically we define $I_k(\nu_i)$ to be 1 if the continuation of ν_i intersects the interior of*

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exactly k of the original polygons not counting ν_i itself. Also the sums involving the original line segments are doubled. For example if all of the polygons in Example 2.3.1 are replaced by line segments the formula for the measure of the set K of planes intersecting exactly k of the line segments for $1 \leq k \leq n$ is

$$m(K) = 2\sum_i \left(-I_k(\nu_i) + I_{k-1}(\nu_i)\right)|\nu_i|$$

$$+\sum_i \left(I_k(d_i) - 2I_{k-1}(d_i) + I_{k-2}(d_i)\right)|d_i|$$

$$+\sum_i \left(-I_k(s_i) + 2I_{k-1}(s_i) - I_{k-2}(s_i)\right)|s_i| .$$

Example 2.3.6 *Consider two squares shown in Figure 2.1. The first square has vertices $A(0,0)$, $B(0,1)$, $C(-1,1)$, and $D(-1,0)$. The second square has vertices $E(2,0)$, $F(2,-1)$, $G(3,-1)$, and $H(3,0)$. Ambartzumian's Theorem 2.1.4 and the examples above cannot be applied to these squares because there are four collinear vertices. Therefore we use the generalization Theorem 2.2.2 to compute the measure of lines separating the two cubes. Side AB is bounded by a separating atom which puts A and B in the same half-plane. Thus the coefficient of the length of AB will be -1. Likewise the coefficient of the length of EF will be negative. The line segment AE is bounded by a separating atom that puts A and E in different half-planes. Thus the coefficient of the length of AE will be $+1$. Likewise the coefficient of the length of BF will be $+1$. None of the other coefficients are bounded by separating atoms. Thus all other*

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coefficients are zero. Thus the measure of lines separating the two squares is $AE + BF - AB - EF$. *This agrees with Crofton's Theorem* (Crofton 1868).

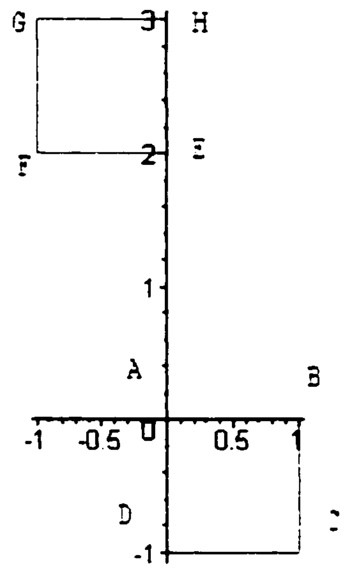

Figure 2.1: Two Squares

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CHAPTER 3

MEASURE OF PLANES SEPARATING POINTS IN R^3

3.1 Results of Ambartzumian and Others on Planes in R^3

Some of Ambartzumian's formulas for computing the measure of planes partitioning a finite number of points in R^3 are summarized below and extended slightly.

Definition 3.1.1 (Ambartzumian 1990. 111) *Let $P = \{P_i\}_1^n$ denote a finite set of points in R^3 such that no 3 of the points are collinear and $n \geq 2$. Almost every plane in R^3 partitions P into two disjoint sets according to which half-*

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space the points lie in. A set of planes in R^3 which form the same partition of P is called an **atom**. *If neither set of the partition of P is empty the atom is called a* **bounded atom**. *The set of unions of bounded atoms forms a* **Radon ring** $r(P)$.

Remark 3.1.2 *Each plane in R^3 determines two half-spaces. Two points $P_i, P_j \in R^3$ may be used to assign a sign to almost all half-spaces in a piecewise continuous way as follows. Put a number scale on the line L through P_i and P_j. Assign a + sign to half-spaces which contain points of L with arbitrarily large values. Assign a − sign to half-spaces which contain points of L with arbitrarily small values.*

Definition 3.1.3 (Ambartzumian 1990, 111) *Let $P = \{P_i\}_1^n$ denote a finite set of points in R^3 such that no 3 of the points are collinear and $n \geq 3$. A* **wedge** *w_s consists of a pair (ν_s, V_s) where ν_s is a line segment whose endpoints are elements of $\{P_i\}_1^n$ and whose interior points are not elements of $\{P_i\}_1^n$ and where V_s is an open domain in R^3 containing no elements of $\{P_i\}_1^n$ and bounded by two planes each of which contain the line segment ν_s and at least one additional point of $\{P_i\}_1^n$ not on the same line as the line segment ν_s. The line segment ν_s is called the* **needle** *of the wedge. In the event that all of the points of $\{P_i\}_1^n$ are coplanar one has a degenerate wedge consisting of the two half-spaces determined by the plane containing $\{P_i\}_1^n$.*

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To each wedge w_s we associate a set of planes as follows which by abuse of notation is also refered to as the wedge w_s. Let the line l_s denote the extension of the line segment ν_s associated with the wedge w_s. A plane will be considered to be an element of a wedge w_s if it is a subset of $V_s \cup l_s$ where V_s is the open domain of R^3 associated with the wedge w_s.

Notation 3.1.4 (Ambartzumian 1990. 112) *Let $P = \{P_i\}_1^n$ denote a finite set of points in R^3 such that no 3 of the points are collinear and $n \geq 3$. Each wedge associated with a needle $\overline{P_i, P_j}$ where $P_i, P_j \in P$ may be used to assign an orientation to planes in neighboring atoms as follows. First arbitrarily assign a + sign to one of the half-spaces determined by a plane of the wedge and a − sign to the other half-space. Next infinitessimally displace the plane away from the wedge into a neighboring atom. There are up to four such atoms corresponding to the four different ways of assigning +/− signs to P_i and P_j. Denote the displacements of P_i and P_j as P_i' and P_j' respectively. If P_i' is in the positive half-space determined by the pair P_i, P_j then designate the half-space containing P_i determined by the displaced plane with a positive sign. Otherwise designate the half-space containing P_i determined by the displaced plane with a negative sign. Extend the orientation continuously to the other planes of the atom. Do this for all four bounding atoms. Let $(\overset{+}{i}, \overset{+}{j})$ denote the bounding atom whose positive half-spaces contain both points P_i and P_j and whose planes*

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otherwise form the same partition of P as a plane of the wedge. Likewise let $(\overset{-}{i},\overset{+}{j})$, $(\overset{+}{i},\overset{-}{j})$, and $(\overset{-}{i},\overset{-}{j})$ *respectively denote atoms whose designated half-spaces contain points* P_i *and* P_j.

Theorem 3.1.5 (Ambartzumian 1990, 112-113) *Let* $P = \{P_i\}_1^n$ *denote a finite set of points in* R^3 *such that no 3 of the points are collinear and* $n \geq 3$. *For each wedge with needle* $P_i, P_j \in P$ *let* $(\overset{+}{i},\overset{+}{j})$, $(\overset{-}{i},\overset{+}{j})$, $(\overset{+}{i},\overset{-}{j})$, and $(\overset{-}{i},\overset{-}{j})$ *respectively denote bounding atoms whose designated half-spaces contain points* P_i *and* P_j. *Let* Q *be an element of the Radon ring of* P. *Let* $I_Q(A)$ *be an indicator function on atoms which takes on the value 1 if* $A \subset Q$ *and 0 otherwise. Let*

$$c_s(Q) = I_Q(\overset{+}{i}.\overset{-}{j}) + I_Q(\overset{-}{i},\overset{+}{j}) - I_Q(\overset{+}{i}.\overset{+}{j}) - I_Q(\overset{-}{i}.\overset{-}{j}).$$

Let $|\nu_s|$ *be the length of the needle* $\overline{P_iP_j}$. *Let* $|V_s|$ *be the angle of the wedge. Let* $m(Q)$ *denote the motion invariant measure on the set of planes in* R^3. *Then*

$$m(Q) = \frac{1}{2}\sum_s c_s(Q)|\nu_s||V_s|.$$

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3.2 A Slight Generalization of Ambartzumian's Results on Planes in R^3

In order to compute the measure of planes separating polyhedra it is useful to have a slightly more general formula that allows for the possibility that three or more points will be collinear.

Definition 3.2.1 *Let* $P = \{P_i\}_1^n$ *denote a finite set of points in* R^3 *not all collinear where* $n \geq 3$ *as above but now we allow for the possibility that three or more points are collinear. Let* $P_i, P_j \in P$. *Then the line segment* $\overline{P_iP_j}$ *is called an* **allowable needle** *if no point of* P *lies on the interior of the line segment. A wedge with an allowable needle is called an* **allowable wedge**. *Given a wedge with an allowable needle* $\overline{P_iP_j}$ *there are up to four* **allowable bounding atoms** *as follows. The two atoms which separate* P_i *and* P_j *and which otherwise induce the same partition of* P *as the wedge are allowable. Also the two atoms for which all points collinear with* P_i *and* P_j *are in the same half-space and which otherwise induce the same partition of* P *as the wedge are allowable.*

A **wedge cluster** *is a set of allowable wedges which share the same open domain and whose needles are line segments on the same line. Each wedge of the wedge cluster is called a* **clustered wedge**. *A* **solitary wedge** *is an*

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allowable wedge whose needle is not collinear with any points from $\{P_i\}_1^n$ *other than its endpoints.*

Theorem 3.2.2 *Let* $P = \{P_i\}_1^n$ *denote a finite set of points in* R^3 *not all collinear for* $n \geq 3$. *For each wedge with an allowable needle* $\overline{P_iP_j}$ *let* $(\overset{+}{i}, \overset{+}{j})$ $(\overset{-}{i}, \overset{+}{j})$, $(\overset{+}{i}, \overset{-}{j})$, and $(\overset{-}{i}, \overset{-}{j})$ *respectively denote the allowable bounding atoms whose designated half-spaces contain points* P_i *and* P_j. *Let* Q *be an element of the Radon ring of* P. *Let* $I_Q(A)$ *be an indicator function on atoms which takes on the value 1 if* $A \subset Q$ *and 0 otherwise. Let*

$$c_s(Q) = I_Q(\overset{+}{i}, \overset{-}{j}) + I_Q(\overset{-}{i}, \overset{+}{j}) - I_Q(\overset{+}{i}, \overset{+}{j}) - I_Q(\overset{-}{i}, \overset{-}{j}).$$

Let $|\nu_s|$ *be the length of the needle* $\overline{P_iP_j}$. *Let* $|V_s|$ *be the angle of the wedge. Let* $m(Q)$ *denote the motion invariant measure on the set of planes in* R^3. *Then*

$$m(Q) = \frac{1}{2}\sum_s c_s(Q)|\nu_s||V_s|.$$

Proof. We will use induction on the number of points. If the set P contains only three points then no three points are collinear and the theorem is true by Theorem 3.1.5. Now suppose that the theorem is true if the set P contains k points. We want to prove that it is true if the set P contains $k+1$ points.

Case I. No three points are collinear. Then the theorem is true by Theorem 3.1.5.

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Case II. P contains a set of at least three collinear points. Without loss of generality, order three of the points of the collinear set P_1, P_2, and P_3. assume that no points of P are interior to the line segment $\overline{P_1P_2}$, and that P_3 is not the interior to a line segment with endpoints in P and collinear with $\overline{P_1P_2}$.

We consider first an atom a_1 that separates two points of a set of three or more collinear points. Without loss of generality assume that the atom a_1 separates P_1 and P_2. Extend the atom a_1 to an atom a_1' of $P\backslash P_3$ by including in the new atom planes that either do or do not separate P_1 and P_3. But any plane that separates P_1 and P_2 will also separate P_1 and P_3. Thus the original atom a_1 of P contains exactly the same set of planes as the new atom a_1' of $P\backslash P_3$.

By supposition the theorem is true on $P\backslash P_3$. Thus it will suffice to show that the coefficient of any wedge which has P_3 as an endpoint is zero. But any allowable bounding atom for P_3 would contain planes which intersect the line through $\overline{P_1P_2}$ arbitrarily close to P_3. Thus any such allowable bounding atom would not separate P_1 and P_2. Thus the indicator function for such a bounding atom at a_1 would be zero. Thus the coefficient of the wedge would be zero.

We next consider an atom a_2 that does not separate any points of a set of three or more collinear points. Let P_1, P_2, and P_3 be three such consecutive

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points as above. Then a_2 is also an atom of $P \backslash P_2$. By supposition the theorem is true on $P \backslash P_2$. Given a wedge with needle $\overline{P_1 P_3}$ in $P \backslash P_2$ there are four possible allowable bounding atoms. Two of these atoms will intersect the needle and thus have coefficient zero. The other two will be bounding atoms for $\overline{P_1 P_2}$ and $\overline{P_2 P_3}$ in P and will have the same coefficient in either ring. Thus the measure of a_2 will be the same in either ring. □

3.3 Examples

Example 3.3.1 *Let $\{D_i\}_1^n$ be a set of bounded convex polyhedral domains with disjoint closures in R^3 with the property that no three vertices are collinear. The measure of the set K of planes intersecting exactly k of the polyhedra for $1 \leq k \leq n$ is*

$$m(K) = \sum_i (-I_k(\nu_i) + I_{k-1}(\nu_i))\,|\nu_i|$$
$$+ \sum_i (I_k(d_i) - 2I_{k-1}(d_i) + I_{k-2}(d_i))\,|d_i|$$
$$+ \sum_i (-I_k(s_i) + 2I_{k-1}(s_i) - I_{k-2}(s_i))\,|s_i|$$

where I_k or $I_k(\cdot)$ denotes the indicator function on the set of wedges which intersect the interiors of exactly k of the polyhedral domains. ν_i represent the original edges of the polyhedra, d_i represent line segments connecting vertices of two different polyhedra and whose wedges separate the two polyhedra. and

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s_i represent line segments connecting vertices of two different polyhedra and whose wedges send both polyhedra to the same half-space.

Proof. Note that if a plane of a wedge intersects the interior of a polyhedron a small perturbation of the plane will still intersect the interior of that polyhedron by properties of open sets. Likewise if a plane of a wedge does not intersect the closure of a polyhedron then a small enough perturbation of the plane will not intersect the closure of a polyhedron. Thus all four bounding atoms will intersect the interiors of exactly the same polyhedra as the planes of the wedge except perhaps those polyhedra with a vertex of the wedge and whose interiors are disjoint from planes of the wedge.

First consider the coefficients of inner wedges on the original edges of the polyhedra. A plane of an inner wedge through a vertex of a polyhedron by definition intersects the interior of that polyhedron. By Remark 3.1.2 above all four of the bounding atoms intersect exactly the same set of polyhedra. Thus the four indicator functions are either all ones or all zeros. Thus the coefficient of I_j on a wedge of this type is either $1+1-1-1$ or $0+0-0-0=0$ for all j.

Next consider the coefficients of half-inner wedges, that is a wedge with vertices on two different polyhedra which intersects the interior of one of these polyhedra say A but not the other one say B. By Remark 3.1.2 above all four

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bounding atoms will intersect the interior of exactly the same set of polyhedra except perhaps B. Let a_i be the vertex at A and let b_i be the vertex at B. An atom which sends a_i and b_i to the same half-space either will or will not separate b_i from the other vertices of B. Likewise an atom which separates a_i and b_i either will or will not separate b_i from the other vertices of B. Thus a coefficient of I_j on a wedge of this type will either be $1 + 0 - 1 - 0$ or $0 + 0 - 0 - 0 = 0$ for all j.

Next consider the coefficients of outer wedges on original edges of the polyhedra. The bounding atom that sends the vertices of the edge to the same half-space as the rest of the vertices of the polyhedron will not intersect the interior of the polyhedron. The other three bounding atoms will separate at least one of the vertices of the polyhedron from the rest of the polyhedron. Thus the coefficient of I_k will be $0 + 0 - 0 - 1 = -1$ and the coefficient of I_{k-1} will be $1 + 1 - 1 - 0 = 1$ on wedges of this type.

Now consider the coefficients of wedges whose vertices are on two different polyhedra and which separate those two polyhedra. We will refer to wedges of this type as **separating wedges**. Note that the bounding atom which sends each vertex of the wedge to the half-space with the other vertices from the same polyhedron intersects neither of the two polyhedra. Note also that the bounding atom which sends each vertex of the wedge to the opposite half-space intersects the interior of both of the polyhedra. Finally there are two bounding

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atoms which send both vertices to the same half-space. Such bounding atoms intersect the interior of exactly one of these two polyhedra. Thus the coefficient of I_k will be $1+0-0-0=1$. the coefficient of I_{k-2} will be $0+1-0-0=1$. and the coefficient of I_{k-1} will be $0+0-1-1=-2$ on wedges of this type.

Finally consider outer wedges with vertices on two polyhedra which send both of these polyhedra to the same half-space. We will refer to these wedges as **linking wedges**. There is one bounding atom that sends both vertices of the wedge to the half-space which contains the other vertices of the two polyhedra. Such an atom will intersect neither polyhedron. There is one bounding atom that will send both vertices of the wedge to the opposite half-space. Such an atom will intersect the interiors of both polyhedra. The other two bounding atoms send exactly one of the vertices of the wedge to the same half-space as the other vertices of the two polyhedra. Such a bounding atom will intersect the interiors of exactly one of these two polyhedra. Thus the coefficient of I_k will be $0+0-1-0=-1$. the coefficient of I_{k-2} will be $0+0-0-1=-1$. and the coefficient of I_{k-1} will be $1+1-0-0=2$ on wedges of this type.

Thus the types of wedges that will have nonzero coefficients are outer wedges on edges. separating wedges, and linking wedges. Thus for $1 \leq k \leq n$ the measure of the planes hitting exactly k of the n polyhedra is

$$\sum_i (-I_k(\nu_i) + I_{k-1}(\nu_i))\,|\nu_i|$$

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$$+\sum_i (I_k(d_i) - 2I_{k-1}(d_i) + I_{k-2}(d_i))\,|d_i|$$

$$+\sum_i (-I_k(s_i) + 2I_{k-1}(s_i) - I_{k-2}(s_i))\,|s_i|$$

where $\{\nu_i\}$ is the set of wedges on edges. $\{d_i\}$ is the set of separating wedges. and $\{s_i\}$ is the set of linking wedges. □

Example 3.3.2 *We can extend the formula of Example 3.3.1 to the measure of planes which intersect the convex hull of the set of polyhedra but none ($k = 0$) of the polyhedra if we include only wedges whose needles intersect the interior of the convex hull in the summations.*

Example 3.3.3 *If $k = n$ the formula of Example 3.3.1 specializes to*

$$\sum_i I_{n-1}(\nu_i)\,|\nu_i| + \sum_i I_{n-2}(d_i)\,|d_i| + \sum_i I_{n-2}(s_i)\,|s_i|\,.$$

This represents the measure of planes hitting all n of the polyhedra.

Example 3.3.4 *The measure of planes hitting at least one of the polyhedra may be obtained by summing the formula of Example 3.3.1 over k for $k = 1 \ldots n$. Much cancellation occurs and the result is*

$$\sum_i I_o(\nu_i)\,|\nu_i| - \sum_i I_o(d_i)\,|d_i| + \sum_i I_o(s_i)\,|s_i|\,.$$

Example 3.3.5 *The more general formula of Theorem 3.2.2 can be applied to parallel cubes of Section 6.2.*

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CHAPTER 4

SEPARATING DOUBLE SUPPORT

4.1 Introduction

In 1869 M.W. Crofton showed that the measure of lines intersecting both members of a pair of disjoint compact convex bodies in R^2 is equal to the length of a taut crossed string enclosing the two bodies minus the perimeters of the two bodies. His proof seemed to assume the existence and uniqueness of the string. Another author on the subject who seemed to make the same assumption was J.J.Sylvester (1890). A search of the literature did not uncover a proof of the existence and uniqueness of the crossed string.

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The crossed string contains a linear part and a part that follows the boundary of the convex bodies. The linear part of the crossed string consists of segments of a pair of lines that are support lines to both convex bodies and which separate the interiors of the two bodies. In this chapter the existence and uniqueness of this pair of separating double support lines for a pair of disjoint compact convex bodies in R^2 will be shown. The results will then be extended to pairs of disjoint compact convex bodies in R^3.

4.2 Containment Half-Planes in R^2

Theorem 4.2.1 *Let A and B be disjoint compact convex bodies in R^2. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Without loss of generality assume that A contains the origin and B contains c. Extend the z-axis to an xz coordinate system with positive orientation. Then A is contained in the closed half-plane $\{(x, z) \in R^2 : z \leq 0\}$ and B is contained in the closed half-plane $\{(x, z) \in R^2 : z \geq c\}$.*

Proof. Let $(x_1, z_1) \in B$ minus the point $(0, c)$.

Case I. Assume that $x_1 > 0$ and $z_1 < c$. Then using basic linear algebra the equation of the line through (x_1, z_1) and $(0, c)$ is $z = \frac{z_1 - c}{x_1}x + c$. Furthermore the equation of the line through the origin and perpendicular to the above line

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is $z = \frac{x_1}{c-z_1}x$. The two lines meet when $x = \frac{c}{\frac{x_1}{c-z_1}+\frac{c-z_1}{x_1}}$. Thus the x-coordinate of the intersection will be positive. Call this point of intersection (x_2, z_2).

Thus the points $(0, c)$, (x_1, z_1), and (x_2, z_2) are collinear. The point $(0, c)$ cannot be an interior point on the line segment containing the other two points since x_1 and x_2 are both positive. The point (x_2, z_2) also cannot be an interior point on the line segment containing the other two points. Otherwise the convexity of B would imply that $(x_2, z_2) \in B$ and perpendicularity would imply that it is closer to the origin than the point $(0, c)$. Furthermore the point (x_1, z_1) cannot be an interior point on the line segment containing the other two points. Otherwise the Pythagorean Theorem would imply that (x_1, z_1) is closer to the origin than $(0, c)$. Thus case I cannot occur.

Case II. Assume that $x_1 = 0$ and $0 < z_1 < c$. Then (x_1, z_1) is on the line segment from the origin to $(0, c)$ and thus is closer to the origin than the point $(0, c)$ Thus Case II cannot occur.

Case III. Assume that $x_1 = 0$ and $z_1 \leq 0$. Then by convexity B contains the origin and thus intersects A. Thus Case III cannot occur.

Case IV. Assume that $x_1 < 0$ and $z_1 < c$. This case cannot occur by an argument analogous to the argument in Case I.

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Cases I through IV above exhaust all of the possibilities for $z_1 < c$ and none of these cases can happen. Thus $z_1 \geq c$. Thus the point (x_1, z_1) and all points of B are in the closed half-plane $\{(x, z) \in R^2 : z \geq c\}$. By a similar argument A is contained in the closed half-plane $\{(x, z) \in R^2 : z \leq 0\}$. □

4.3 Signed Distance

Definition 4.3.1 *The* **signed distance** *between nondegenerate finite collinear line segments is defined as follows. If the line segments overlap then the distance is the negative of the length of the intersection. If the line segments are disjoint then the distance is the length of the shortest line segment connecting them.*

The signed distance can be regarded as a function of the endpoints of the two line segments. Let $A = [a_1, a_2]$ *with* $a_1 < a_2$ *and* $B = [b_1, b_2]$ *with* $b_1 < b_2$ *be collinear line segments. Then the signed distance regarded as a map from*

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$\{(a_1, a_2, b_1, b_2) \in R^4 : a_1 < a_2, b_1 < b_2\}$ *to* R *is given by the formula*

$$(4.3.2) \qquad d(a_1, a_2, b_1, b_2) = \begin{cases} b_1 - a_2 & \text{if} \quad a_1 < a_2 \le b_1 < b_2 \\ b_1 - a_2 & \text{if} \quad a_1 \le b_1 \le a_2 \le b_2 \\ b_1 - b_2 & \text{if} \quad a_1 \le b_1 < b_2 \le a_2 \\ a_1 - b_2 & \text{if} \quad b_1 < b_2 \le a_1 < a_2 \\ a_1 - b_2 & \text{if} \quad b_1 \le a_1 \le b_2 \le a_2 \\ a_1 - a_2 & \text{if} \quad b_1 \le a_1 < a_2 \le b_2 \end{cases}.$$

Theorem 4.3.3 *Given two collinear line segments* $A = [a_1, a_2]$ *with* $a_1 < a_2$ *and* $B = [b_1, b_2]$ *with* $b_1 < b_2$ *the signed distance between them is continuous.*

A Short Proof. The signed distance is piecewise linear and the formulas agree on the boundaries. Therefore signed distance is continuous. □

A δ-ϵ Proof. Let D be the domain of the signed distance function.

Thus $D = \{(a_1, a_2, b_1, b_2) \in R^4 : a_1 < a_2, b_1 < b_2\}$.

Let $\delta_1 = \min\left(\frac{a_2 - a_1}{2}, \frac{b_2 - b_1}{2}\right)$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta_1$.

Then $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta_1$

implies $|a_3 - a_1| < \frac{a_2 - a_1}{2}$ and $|a_4 - a_2| < \frac{a_2 - a_1}{2}$

which implies $a_3 - a_1 < \frac{a_2 - a_1}{2}$ and $\frac{a_1 - a_2}{2} < a_4 - a_2$

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which implies $a_3 < \frac{a_1+a_2}{2}$ and $\frac{a_1+a_2}{2} < a_4$

which implies $a_3 < a_4$

Since the formulas are symmetric in a and b then also $b_3 < b_4$. Thus $(a_3, a_4, b_3, b_4) \in D$ the domain of the distance function and the above formula 4.3.2 for the distance function can be applied replacing subscript 1 with subscript 3 and replacing subscript 2 with subscript 4 to yield the formula:

$$(4.3.4) \qquad d(a_3, a_4, b_3, b_4) = \begin{cases} b_3 - a_4 & \text{if} \quad a_3 < a_4 \leq b_3 < b_4 \\ b_3 - a_4 & \text{if} \quad a_3 \leq b_3 \leq a_4 \leq b_4 \\ b_3 - b_4 & \text{if} \quad a_3 \leq b_3 < b_4 \leq a_4 \\ a_3 - b_4 & \text{if} \quad b_3 < b_4 \leq a_3 < a_4 \\ a_3 - b_4 & \text{if} \quad b_3 \leq a_3 \leq b_4 \leq a_4 \\ a_3 - a_4 & \text{if} \quad b_3 \leq a_3 < a_4 \leq b_4 \end{cases}$$

In what follows the above formulas will be used to show that the signed distance function d is continuous by showing that given an arbitrary $\epsilon > 0$ there is a $\delta > 0$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$ implies $|d(a_1, a_2, b_1, b_2) - d(a_3, a_4, b_3, b_4)| < \epsilon$.

Because the signed distance function is defined piecewise the proof will be broken down into cases. Furthermore since the formula for signed distance is symmetric in a and b one can eliminate some cases by assuming without loss of generality that $a_1 \leq b_1$. Therefore only the first three pieces of the piecewise

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defined function in formula 4.3.4 need be considered. Thus only the following cases need be considered.

Case I. $a_1 < b_1$ and $a_2 < b_2$

Then $d(a_1, a_2, b_1, b_2) = b_1 - a_2$.

Let $\epsilon > 0$. Let $\delta = \min\left(\delta_1, \frac{\epsilon}{2}, \frac{b_1-a_1}{2}, \frac{b_2-a_2}{2}\right)$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$.

Then $|a_3 - a_1| < \frac{b_1-a_1}{2}$ and $|b_3 - b_1| < \frac{b_1-a_1}{2}$

Thus $a_3 - a_1 < \frac{b_1-a_1}{2}$ and $\frac{a_1-b_1}{2} < b_3 - b_1$

Therefore $a_3 < \frac{a_1+b_1}{2}$ and $\frac{a_1+b_1}{2} < b_3$

which implies $a_3 < b_3$

Since the formulas are symmetric in a and b then also $a_4 < b_4$

The two inequalities $a_3 < b_3$ and $a_4 < b_4$ together imply that

$d(a_3, a_4, b_3, b_4) = b_3 - a_4$

Thus $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - a_4) - (b_1 - a_2)|$

$= |(b_3 - b_1) + (a_2 - a_4)| \leq |b_3 - b_1| + |a_2 - a_4| < 2\delta \leq \epsilon$

Case II. $a_1 = b_1 < a_2 = b_2$

Then $d(a_1, a_2, b_1, b_2) = b_1 - a_2 = b_1 - b_2 = a_1 - a_2$.

Let $\epsilon > 0$. Let $\delta = \min\left(\delta_1, \frac{\epsilon}{2}\right)$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$.

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Note that there are four distinct formulas given above for the signed distance evaluated at (a_3, a_4, b_3, b_4). Each formula will be considered separately below.

Subcase IIa $d(a_3, a_4, b_3, b_4) = b_3 - a_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - a_4) - (b_1 - a_2)|$

$= |(b_3 - b_1) + (a_2 - a_4)| \leq |b_3 - b_1| + |a_2 - a_4| < 2\delta \leq \epsilon$

Subcase IIb $d(a_3, a_4, b_3, b_4) = b_3 - b_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - b_4) - (b_1 - b_2)|$

$= |(b_3 - b_1) + (b_2 - b_4)| \leq |b_3 - b_1| + |b_2 - b_4| < 2\delta \leq \epsilon$

Subcase IIc $d(a_3, a_4, b_3, b_4) = a_3 - b_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(a_3 - b_4) - (a_1 - b_2)|$

$= |(a_3 - a_1) + (b_2 - b_4)| \leq |a_3 - a_1| + |b_2 - b_4| < 2\delta \leq \epsilon$

Subcase IId $d(a_3, a_4, b_3, b_4) = a_3 - a_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(a_3 - a_4) - (a_1 - a_2)|$

$= |(a_3 - a_1) + (a_2 - a_4)| \leq |a_3 - a_1| + |a_2 - a_4| < 2\delta \leq \epsilon$

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Case III. $a_1 = b_1 < a_2 < b_2$

Then $d(a_1, a_2, b_1, b_2) = b_1 - a_2 = a_1 - a_2$.

Let $\epsilon > 0$. Let $\delta = \min\ (\delta_1, \frac{\epsilon}{2}, \frac{a_2-b_1}{2}, \frac{b_2-a_2}{2})$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$.

Then $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$

implies $|a_2 - a_4| < \frac{b_2-a_2}{2}$ and $|b_2 - b_4| < \frac{b_2-a_2}{2}$

which implies $a_4 - a_2 < \frac{b_2-a_2}{2}$ and $\frac{a_2-b_2}{2} < b_4 - b_2$

which implies $a_4 < \frac{a_2+b_2}{2}$ and $\frac{a_2+b_2}{2} < b_4$

which implies $a_4 < b_4$

which implies $d(a_3, a_4, b_3, b_4) = b_3 - a_4$ or $a_3 - a_4$

Subcase IIIa $d(a_3, a_4, b_3, b_4) = b_3 - a_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - a_4) - (b_1 - a_2)|$

$= |(b_3 - b_1) + (a_2 - a_4)| \leq |b_3 - b_1| + |a_2 - a_4| < 2\delta \leq \epsilon$

Subcase IIIb $d(a_3, a_4, b_3, b_4) = a_3 - a_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(a_3 - a_4) - (a_1 - a_2)|$

$= |(a_3 - a_1) + (a_2 - a_4)| \leq |a_3 - a_1| + |a_2 - a_4| < 2\delta \leq \epsilon$

Case IV. $a_1 < b_1 < a_2 = b_2$

Then $d(a_1, a_2, b_1, b_2) = b_1 - a_2 = b_1 - b_2$.

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Let $\epsilon > 0$. Let $\delta = \min\ (\delta_1, \frac{\epsilon}{2}, \frac{b_1-a_1}{2}, \frac{a_2-b_1}{2})$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$.

Then $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$

implies $|a_3 - a_1| < \frac{b_1-a_1}{2}$ and $|b_3 - b_1| < \frac{b_1-a_1}{2}$

which implies $a_3 - a_1 < \frac{b_1-a_1}{2}$ and $\frac{a_1-b_1}{2} < b_3 - b_1$

which implies $a_3 < \frac{a_1+b_1}{2}$ and $\frac{a_1+b_1}{2} < b_3$

which implies $a_3 < b_3$

which implies $d(a_3, a_4, b_3, b_4) = b_3 - a_4$ or $b_3 - b_4$

Subcase IVa $d(a_3, a_4, b_3, b_4) = b_3 - a_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - a_4) - (b_1 - a_2)|$

$= |(b_3 - b_1) + (a_2 - a_4)| \leq |b_3 - b_1| + |a_2 - a_4| < 2\delta \leq \epsilon$

Subcase IVb $d(a_3, a_4, b_3, b_4) = b_3 - b_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - b_4) - (b_1 - b_2)|$

$= |(b_3 - b_1) + (b_2 - b_4)| \leq |b_3 - b_1| + |b_2 - b_4| < 2\delta \leq \epsilon$

Case V. $a_1 = b_1 < b_2 < a_2$

Then $d(a_1, a_2, b_1, b_2) = b_1 - b_2 = a_1 - b_2$.

Let $\epsilon > 0$. Let $\delta = \min\ (\delta_1, \frac{\epsilon}{2}, \frac{b_2-b_1}{2}, \frac{a_2-b_2}{2})$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$.

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Then $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$

implies $|b_4 - b_2| < \frac{a_2-b_2}{2}$ and $|a_4 - a_2| < \frac{a_2-b_2}{2}$

which implies $b_4 - b_2 < \frac{a_2-b_2}{2}$ and $\frac{b_2-a_2}{2} < a_4 - a_2$

which implies $b_4 < \frac{a_2+b_2}{2}$ and $\frac{a_2+b_2}{2} < a_4$

which implies $b_4 < a_4$

which implies $d(a_3, a_4, b_3, b_4) = a_3 - b_4$ or $b_3 - b_4$

Subcase Va $d(a_3, a_4, b_3, b_4) = a_3 - b_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(a_3 - b_4) - (a_1 - b_2)|$

$= |(a_3 - a_1) + (b_2 - b_4)| \leq |a_3 - a_1| + |b_2 - b_4| < 2\delta \leq \epsilon$

Subcase Vb $d(a_3, a_4, b_3, b_4) = b_3 - b_4$

Then $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - b_4) - (b_1 - b_2)|$

$= |(b_3 - b_1) + (b_2 - b_4)| \leq |b_3 - b_1| + |b_2 - b_4| < 2\delta \leq \epsilon$

Case VI. $a_1 < b_1 < b_2 < a_2$

Then $d(a_1, a_2, b_1, b_2) = b_1 - b_2$.

Let $\epsilon > 0$. Let $\delta = \min\ (\delta_1, \frac{\epsilon}{2}, \frac{b_1-a_1}{2}, \frac{b_2-b_1}{2}, \frac{a_2-b_2}{2})$.

Let $(a_3, a_4, b_3, b_4) \in R^4$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$.

Then $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$

implies $|a_3 - a_1| < \frac{b_1-a_1}{2}$ and $|b_3 - b_1| < \frac{b_1-a_1}{2}$

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which implies $a_3 - a_1 < \frac{b_1 - a_1}{2}$ and $\frac{a_1 - b_1}{2} < b_3 - b_1$

which implies $a_3 < \frac{a_1 + b_1}{2}$ and $\frac{a_1 + b_1}{2} < b_3$

which implies $a_3 < b_3$

Also $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$

$|b_4 - b_2| < \frac{a_2 - b_2}{2}$ and $|a_4 - a_2| < \frac{a_2 - b_2}{2}$

which implies $b_4 - b_2 < \frac{a_2 - b_2}{2}$ and $\frac{b_2 - a_2}{2} < a_4 - a_2$

which implies $b_4 < \frac{a_2 + b_2}{2}$ and $\frac{a_2 + b_2}{2} < a_4$

which implies $b_4 < a_4$

The two inequalities $a_3 < b_3$ and $b_4 < a_4$ together imply that

$d(a_3, a_4, b_3, b_4) = b_3 - b_4$

Thus $|d(a_3, a_4, b_3, b_4) - d(a_1, a_2, b_1, b_2)| = |(b_3 - b_4) - (b_1 - b_2)|$

$= |(b_3 - b_1) + (b_2 - b_4)| \leq |b_3 - b_1| + |b_2 - b_4| < 2\delta \leq \epsilon$

The first piece of the piecewise function defined in formula 4.3.2 is covered by Case I. The second piece is covered by Cases I, II, III, and IV. The third piece is covered by Cases II, IV, V, and VI. The last three pieces are covered by the symmetry of the formula in a and b. Thus the six cases mentioned above are exhaustive.

Thus in each case given an arbitrary $\epsilon > 0$ one can find a $\delta > 0$ such that $|(a_1, a_2, b_1, b_2) - (a_3, a_4, b_3, b_4)| < \delta$ implies $|d(a_1, a_2, b_1, b_2) - d(a_3, a_4, b_3, b_4)| < \epsilon$. Therefore the signed distance function d is continuous. □

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4.4 Separation Function

Definition 4.4.1 *Let A and B be disjoint compact convex sets in R^2. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Let $\phi \in R$. Let $L(\phi)$ be a rotation of the z-axis by angle ϕ. Then the* **separation function** *of (A, B, ϕ) is the signed distance between the orthogonal projections of A and B onto $L(\phi)$.*

Lemma 4.4.2 *Let A and B be disjoint compact convex bodies in R^2. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Let $\phi, \phi_0 \in R$. Let $L(\phi)$ be a rotation of the z-axis by angle ϕ. Let A_{0C} be a rectangle circumscribing A with two sides perpendicular to $L(\phi_0)$. Let B_{0C} be a rectangle circumscribing B with two sides perpendicular to $L(\phi_0)$. Then the separation function of (A_{0C}, B_{0C}, ϕ) is a continuous function of ϕ at ϕ_0.*

Proof. Without loss of generality assume that $\phi > 0$ corresponds to a clockwise rotation of the z-axis. Write the separation function $f_C = f_{0C}(\phi)$ as a composition of functions as follows. Order the vertices of A_{0C} and B_{0C} so that the first four vertices are vertices of A_{0C} in dictionary order and so that the last

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four vertices are vertices of B_{0C} in the dictionary order. Let $f_1(\phi) : R \to R^8$ be defined as the ordered orthogonal projections of the vertices of A_0 and B_0 onto $L(\varphi)$. Let $f_2 : R^8 \to R^4$ be defined by $f_2(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) = (min(a_i), max(a_i), min(b_i), max(b_i))$. Let $f_3 : R^4 \to R$ be defined as the signed distance between the line segment determined by the first two coordinates and the line segment determined by the second two coordinates.

f_1 is equivalent to rotating the vertices of A_{0C} and B_{0C} by angle φ where $\phi > 0$ corresponds to a counterclockwise rotation and then projecting orthogonally onto the z-axis. Thus for $i = 1, \cdots, 8$ the ith component of $f_1(\varphi)$ is $f_{1i}(\phi) = x_j \sin\phi + z_k \cos\varphi$ where (x_j, z_k) is the ith ordered vertex of the pair (A_{0C}, B_{0C}). Thus $f_{1i}(\phi)$ is continuous in ϕ since it is a linear combination of continuous functions. Therefore f_1 is continuous in φ since its components are.

Next to prove the continuity of f_2 consider first the first component, renumber the input if necessary so that $a_1 \le a_2 \le a_3 \le a_4$, let a denote (a_1, a_2, a_3, a_4), and consider the following cases.

Case I. $a_1 < a_2 \le a_3 < a_4$. Then $f_{21}(a) = a_1$. Let $\epsilon > 0$. Let $\delta = min(\frac{a_2-a_1}{2}, \frac{a_4-a_3}{2}, \epsilon)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be such that $|a - \alpha| < \delta$. Then $\alpha_2 - \alpha_1 = (\alpha_2 - a_2) + (a_2 - a_1) + (a_1 - \alpha_1) > -\delta + 2\delta + \delta = 0$ which implies that $\alpha_1 < \alpha_2$. By a similar arguments $\alpha_1 < \alpha_3$ and $\alpha_1 < \alpha_4$. Thus for $|a - \alpha| <$

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$\delta, f_{21}(\alpha) = \alpha_1$. Thus for $|\alpha - a| < \delta, |f_{21}(\alpha) - f_{21}(a)| = |\alpha_1 - a_1| < \delta \leq \epsilon$. Thus the first component of f_2 is continuous at a.

Case II. $a_1 = a_2 < a_3 < a_4$. Then $f_{21}(a) = a_1 = a_2$. Let $\epsilon > 0$. Let $\delta = min(\frac{a_3-a_2}{2}, \frac{a_4-a_3}{2}, \epsilon)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be such that $|a - \alpha| < \delta$. Then $\alpha_3 - \alpha_i = (\alpha_3 - a_3) + (a_3 - a_i) + (a_1 - \alpha_i) > -\delta + 2\delta + \delta = 0$ which implies that $\alpha_i < \alpha_3$ for $i = 1, 2$. By a similar argument $\alpha_i < \alpha_4$ for $i = 1, 2$. Thus for $|a - \alpha| < \delta, f_{21}(\alpha) =$ either α_1 or α_2. Thus for $|\alpha - a| < \delta, |f_{21}(\alpha) - f_{21}(a)| =$ either $|\alpha_1 - a_1| < \delta \leq \epsilon$ or $|\alpha_2 - a_2| < \delta \leq \epsilon$. Thus the first component of f_2 is continuous at a.

Case III. $a_1 = a_2 < a_3 = a_4$. Then $f_{21}(a) = a_1 = a_2$. Let $\epsilon > 0$. Let $\delta = min(\frac{a_3-a_2}{2}, \epsilon)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be such that $|a - \alpha| < \delta$. Then $\alpha_i - \alpha_j = (\alpha_i - a_i) + (a_i - a_j) + (a_j - \alpha_j) > -\delta + 2\delta + \delta = 0$ which implies that $\alpha_j < \alpha_i$ for $j = 1, 2$ and $i = 3, 4$. Thus for $|a - \alpha| < \delta, f_{21}(\alpha) =$ either α_1 or α_2. Thus for $|\alpha - a| < \delta, |f_{21}(\alpha) - f_{21}(a)| =$ either $|\alpha_1 - a_1| < \delta \leq \epsilon$ or $|\alpha_2 - a_2| < \delta \leq \epsilon$. Thus the first component of f_2 is continuous at a.

Case IV. $a_1 < a_2 < a_3 = a_4$. Then $f_{21}(a) = a_1$. Let $\epsilon > 0$. Let $\delta = min(\frac{a_3-a_2}{2}, \frac{a_2-a_1}{2}, \epsilon)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be such that $|a - \alpha| < \delta$. Then $\alpha_i - \alpha_1 = (\alpha_i - a_i) + (a_i - a_1) + (a_1 - \alpha_1) > -\delta + 2\delta - \delta = 0$ which implies

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that $\alpha_1 < \alpha_i$ for $i = 2, 3, 4$. Thus for $|a - \alpha| < \delta, f_{21}(\alpha) = \alpha_1$. Thus for $|\alpha - a| < \delta, |f_{21}(\alpha) - f_{21}(a)| = |\alpha_1 - a_1| < \delta \leq \epsilon$. Thus the first component of f_2 is continuous at a.

The above cases are exhaustive. Thus the first component of f_2 is continuous at a and therefore continuous. By a similar arguments the other components of f_2 are continuous. Therefore f_2 is continuous. f_3 is continuous by Theorem 4.3.3. Therefore f_C is continuous since it is a composition of continuous functions. □

Lemma 4.4.3 *Let A and B be disjoint compact convex bodies in R^2. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Let $\phi, \phi_0 \in R$. Let $L(\phi)$ be a rotation of the z-axis by angle ϕ. Let A_{0C} be a rectangle circumscribing A with two sides perpendicular to $L(\phi_0)$. Let B_{0C} be a rectangle circumscribing B with two sides perpendicular to $L(\phi_0)$. Let A_{0I} be a line segment inscribed in A with endpoints on opposite sides of A_{0C} such that these opposite sides are perpendicular to $L(\phi_0)$. Let B_{0I} be a line segment inscribed in B with endpoints on opposite sides of B_{0C} such that these opposite*

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sides are perpendicular to $L(\phi_0)$. Then the separation function of (A_{0l}, B_{0l}, ϕ) is a continuous function of ϕ at ϕ_0.

Proof. Without loss of generality assume that $\phi > 0$ corresponds to a clockwise rotation of the z-axis. Write the separation function $f_l = f_{0l}(\phi)$ as a composition of functions as follows. Order the endpoints of A_{0l} and B_{0l} so that the first two endpoints are endpoints of A_{0l} in dictionary order and so that the last two endpoints are endpoints of B_{0l} in the dictionary order. Let $f_1(\phi) : R \to R^4$ be defined as the ordered orthogonal projections of the endpoints of A_{0l} and B_{0l} onto $L(\phi)$. Let $f_3 : R^4 \to R$ be defined as the signed distance between the line segment determined by the first two coordinates and the line segment determined by the second two coordinates. Therefore f_l is continuous since it is a composition of continuous functions. □

Lemma 4.4.4 *Let A and B be disjoint compact convex bodies in R^2. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Let $\phi, \phi_0 \in R$. Let $L(\phi)$ be a rotation of the z-axis by angle ϕ. Let A_{0C} be a rectangle circumscribing A with two sides perpendicular to $L(\phi_0)$. Let B_{0C} be a rectangle circumscribing B with two sides perpendicular to $L(\phi_0)$. Let A_{0l} be a line segment inscribed in A with endpoints on opposite sides of A_{0C}*

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such that these opposite sides are perpendicular to $L(\phi_0)$. *Let* B_{0I} *be a line segment inscribed in* B *with endpoints on opposite sides of* B_{0C} *such that these opposite sides are perpendicular to* $L(\phi_0)$. *Let* $f = f(\phi)$ *denote the separation function of* A *and* B. *Let* $f_C = f_{0C}(\phi)$ *denote the separation function of* A_{0C} *and* B_{0C}. *Let* $f_I = f_{0I}(\phi)$ *denote the separation function of* A_{0I} *and* B_{0I}. *Then* $f_C \leq f \leq f_I$.

Proof. Since the projections of convex sets are convex. the images of the projections of convex sets onto a line will be (possibly degenerate) line segments. Thus one can coordinatize the images of the projections as follows.

$$\text{Let} \quad \begin{cases} \text{Proj}A_{0I} &= [a_{I1}(\phi), a_{I2}(\phi)] = [a_{I1}, a_{I2}] \\ \text{Proj}A &= [a_1(\phi), a_2(\phi)] = [a_1, a_2] \\ \text{Proj}A_{0C} &= [a_{C1}(\phi), a_{C2}(\phi)] = [a_{C1}, a_{C2}] \\ \text{Proj}B_{0I} &= [b_{I1}(\phi), b_{I2}(\phi)] = [b_{I1}, b_{I2}] \\ \text{Proj}B &= [b_1(\phi), b_2(\phi)] = [b_1, b_2] \\ \text{Proj}B_{0C} &= [b_{C1}(\phi), b_{C2}(\phi)] = [b_{C1}, b_{C2}] \end{cases}$$

Applying the separation function formulas to f_I. f. and f_C respectively then

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gives:

$$f_I = \begin{cases} b_{I1} - a_{I2} & \text{if} \quad a_{I1} \leq a_{I2} \leq b_{I1} \leq b_{I2} \\ b_{I1} - a_{I2} & \text{if} \quad a_{I1} \leq b_{I1} \leq a_{I2} \leq b_{I2} \\ b_{I1} - b_{I2} & \text{if} \quad a_{I1} \leq b_{I1} \leq b_{I2} \leq a_{I2} \\ a_{I1} - b_{I2} & \text{if} \quad b_{I1} \leq b_{I2} \leq a_{I1} \leq a_{I2} \\ a_{I1} - b_{I2} & \text{if} \quad b_{I1} \leq a_{I1} \leq b_{I2} \leq a_{I2} \\ a_{I1} - a_{I2} & \text{if} \quad b_{I1} \leq a_{I1} \leq a_{I2} \leq b_{I2} \end{cases} \tag{4.4.5}$$

$$f = \begin{cases} b_1 - a_2 & \text{if} \quad a_1 \leq a_2 \leq b_1 \leq b_2 \\ b_1 - a_2 & \text{if} \quad a_1 \leq b_1 \leq a_2 \leq b_2 \\ b_1 - b_2 & \text{if} \quad a_1 \leq b_1 \leq b_2 \leq a_2 \\ a_1 - b_2 & \text{if} \quad b_1 \leq b_2 \leq a_1 \leq a_2 \\ a_1 - b_2 & \text{if} \quad b_1 \leq a_1 \leq b_2 \leq a_2 \\ a_1 - a_2 & \text{if} \quad b_1 \leq a_1 \leq a_2 \leq b_2 \end{cases} \tag{4.4.6}$$

$$f_C = \begin{cases} b_{C1} - a_{C2} & \text{if} \quad a_{C1} \leq a_{C2} \leq b_{C1} \leq b_{C2} \\ b_{C1} - a_{C2} & \text{if} \quad a_{C1} \leq b_{C1} \leq a_{C2} \leq b_{C2} \\ b_{C1} - b_{C2} & \text{if} \quad a_{C1} \leq b_{C1} \leq b_{C2} \leq a_{C2} \\ a_{C1} - b_{C2} & \text{if} \quad b_{C1} \leq b_{C2} \leq a_{C1} \leq a_{C2} \\ a_{C1} - b_{C2} & \text{if} \quad b_{C1} \leq a_{C1} \leq b_{C2} \leq a_{C2} \\ a_{C1} - a_{C2} & \text{if} \quad b_{C1} \leq a_{C1} \leq a_{C2} \leq b_{C2} \end{cases} \tag{4.4.7}$$

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Also note that $A_{0I} \subseteq A \subseteq A_{0C}$ and $B_{0I} \subseteq B \subseteq B_{0C}$.

Thus Proj $A_{0I} \subseteq$ Proj $A \subseteq$ Proj A_{0C} and Proj $B_{0I} \subseteq$ Proj $B \subseteq$ Proj B_{0C}.

Thus

$$a_{C1} \leq a_1 \leq a_{I1} \leq a_{I2} \leq a_2 \leq a_{C2} \tag{4.4.8}$$

$$b_{C1} \leq b_1 \leq b_{I1} \leq b_{I2} \leq b_2 \leq b_{C2}. \tag{4.4.9}$$

Formulas 4.4.5. 4.4.6. 4.4.7. 4.4.8. and 4.4.9 above can then be applied to each of the cases below.

Case I. $a_1 \leq a_2 \leq b_1 \leq b_2$

Then $a_{I1} \leq a_{I2} \leq b_{I1} \leq b_{I2}$

Thus $f = b_1 - a_2 \leq b_{I1} - a_{I2} = f_I$

Also

$$f_C = \begin{cases} b_{C1} - a_{C2} \leq b_1 - a_2 = f \\ \text{or} \quad b_{C1} - b_{C2} \leq b_1 - b_2 \leq b_1 - a_2 = f \\ \text{or} \quad a_{C1} - b_{C2} \leq a_1 - b_2 \leq b_1 - a_2 = f \\ \text{or} \quad a_{C1} - a_{C2} \leq a_1 - a_2 \leq b_1 - a_2 = f \end{cases}$$

Thus for Case I. $f_C \leq f \leq f_I$

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Case II. $a_1 \leq b_1 \leq a_2 \leq b_2$

Then

$$f_I = \begin{cases} b_{I1} - a_{I2} \geq b_1 - a_2 = f \\ \text{or} \quad b_{I1} - b_{I2} \geq b_{I1} - a_{I2} \geq b_1 - a_2 = f \\ \text{or} \quad a_{I1} - b_{I2} \geq b_{I1} - a_{I2} \geq b_1 - a_2 = f \\ \text{or} \quad a_{I1} - a_{I2} \geq b_{I1} - a_{I2} \geq b_1 - a_2 = f \end{cases}$$

Also

$$f_C = \begin{cases} b_{C1} - a_{C2} \leq b_1 - a_2 = f \\ \text{or} \quad b_{C1} - b_{C2} \leq b_1 - b_2 \leq b_1 - a_2 = f \\ \text{or} \quad a_{C1} - b_{C2} \leq a_1 - b_2 \leq b_1 - a_2 = f \\ \text{or} \quad a_{C1} - a_{C2} \leq a_1 - a_2 \leq b_1 - a_2 = f \end{cases}$$

Thus for Case II. $f_C \leq f \leq f_I$

Case III. $a_1 \leq b_1 \leq b_2 \leq a_2$

Then

$$f_I = \begin{cases} b_{I1} - a_{I2} \geq b_{I1} - b_{I2} =\geq b_1 - b_2 = f \\ \text{or} \quad b_{I1} - b_{I2} \geq b_1 - b_2 = f \\ \text{or} \quad a_{I1} - b_{I2} \geq b_{I1} - b_{I2} \geq b_1 - b_2 = f \\ \text{or} \quad a_{I1} - a_{I2} \geq b_{I1} - b_{I2} \geq b_1 - b_2 = f \end{cases}$$

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Also

$$f_C = \begin{cases} & b_{C1} - a_{C2} \le b_1 - a_2 \le b_1 - b_2 = f \\ \text{or} & b_{C1} - b_{C2} \le b_1 - b_2 = f \\ \text{or} & a_{C1} - b_{C2} \le a_1 - b_2 \le b_1 - b_2 = f \\ \text{or} & a_{C1} - a_{C2} \le a_1 - a_2 \le b_1 - b_2 = f \end{cases}$$

Thus for Case III. $f_C \le f \le f_I$

Case IV. $b_1 \le b_2 \le a_1 \le a_2$

The argument is similar to the argument for Case I but switch the roles of a and b.

Case V. $b_1 \le a_1 \le b_2 \le a_2$

The argument is similar to the argument for Case II but switch the roles of a and b.

Case VI. $b_1 \le a_1 \le a_2 \le b_2$

The argument is similar to the argument for Case III but switch the roles of a and b.

The above six cases are exhaustive. Therefore $f_C \le f \le f_I$. □

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Theorem 4.4.10 *Let A and B be disjoint compact convex bodies in R^2 with positive area. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Let $\phi \in R$. Let $L(\phi)$ be a rotation of the z-axis by angle ϕ. Then the separation function of (A, B, ϕ) is a continuous function of ϕ.*

Proof. Without loss of generality assume that $\phi > 0$ corresponds to a clockwise rotation. Let $\phi_0 \in R$. Let A_{0C} be a rectangle circumscribing A with two sides perpendicular to $L(\phi_0)$. Let B_{0C} be a rectangle circumscribing B with two sides perpendicular to $L(\phi_0)$. Let $f_C = f_{0C}(\phi)$ denote the separation function of A_{0C} and B_{0C}. Let A_{0I} be a line segment inscribed in A with endpoints on opposite sides of A_{0C} such that these opposite sides are perpendicular to $L(\phi_0)$. Let B_{0I} be a line segment inscribed in B with endpoints on opposite sides of B_{0C} such that these opposite sides are perpendicular to $L(\phi_0)$. Let $f_I = f_{0I}(\phi)$ denote the separation function of A_{0I} and B_{0I}. Let $\epsilon > 0$. Then using lemmas 4.4.2, 4.4.3, and 4.4.4 there is $\delta > 0$ such that $|\phi - \phi_0| < \delta$ implies

$$f(\phi) - f(\phi_0) = f(\phi) - f_{0I}(\phi_0) \leq f_{0I}(\phi) - f_{0I}(\phi_0) < \epsilon$$

$$\text{and } f(\phi_0) - f(\phi) = f_{0C}(\phi_0) - f(\phi) \leq f_{0C}(\phi_0) - f_{0C}(\phi) < \epsilon.$$

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Thus $|f(\phi_0) - f(\phi)| < \epsilon$. Thus f is continuous at ϕ_0. Thus f is continuous at ϕ. □

4.5 Examples of Separation Functions

Using elementary trigonometry the separation function is computed for some simple examples below.

Example 4.5.1 *Let A be a circle of radius r centered at $(0. -r)$. Let B be a circle of radius r centered at $(0. c+r)$. Then the separation function*

$$f(A, B. \varphi) = (c + 2r)|\cos\varphi| - 2r$$

Example 4.5.2 *Let A be a square centered at $(0. -s/2)$ whose sides are parallel to the axes and have length s. Let B be a square centered at $(0. c + s/2)$ whose sides are parallel to the axes and have length s. Then the separation function*

$$f(A. B. \varphi) = \begin{cases} c\cos\varphi + s\sin\varphi & \text{if} \quad -\pi/2 \leq \varphi \leq 0 \\ c\cos\varphi + s\sin\varphi & \text{if} \quad 0 \leq \varphi \leq \pi/2 \end{cases}$$

Example 4.5.3 *Let A be a square centered at $(s/2. -s/2)$ whose sides are parallel to the axes and have length s. Let B be a square centered at $(-s/2. c+$*

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$s/2$) whose sides are parallel to the axes and have length s. Then the separation function

$$f(A, B, \phi) = \begin{cases} c\cos\phi & \text{if} \quad -\pi/2 \le \phi \le 0 \\ c\cos\phi - 2s\sin\phi & \text{if} \quad 0 \le \phi \le \tan^{-1}(\frac{c+s}{s}) \\ -(c+2s)\cos\phi\cos\phi & \text{if} \tan^{-1}(\frac{c+s}{s}) \le \phi \le \pi/2 \end{cases}$$

4.6 Separating Double Support Lines in R^2

Lemma 4.6.1 *Let A and B be disjoint compact convex bodies in R^2 with positive area. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Then there is at least one separating double support line whose upward normal vector makes an angle ϕ_1 clockwise from the z-axis such that $0 < \phi_1 \le \pi/2$ and at least one additional distinct separating double support line whose upward normal vector makes an angle ϕ_2 counterclockwise from the z-axis such that $0 < \phi_2 \le \pi/2$.*

Proof. Orthogonally project A and B onto the x-axis. Since A and B are convex then their projections will be line segments. Since the z-axis projects to the origin and intersects both A and B then the signed distance between the projections of A and B will be less than or equal to zero.

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Now orthogonally project A and B onto the z-axis. Then the signed distance between the projections will be $c > 0$ by Theorem 4.2.1.

Now orthogonally project A and B onto a line $L(\phi)$ making angle ϕ clockwise with the positive z-axis where $0 \leq \phi \leq \pi/2$. Then the signed distance between the projections of A and B is a continuous function of ϕ by Theorem 4.4.10.

Thus since the signed distance between the projections of A and B is a continuous function of ϕ which is positive when $\phi = 0$ and nonpositive when $\phi = \pi/2$ then the Intermediate Value Theorem implies that this distance is 0 for some value of ϕ such that $0 < \phi \leq \pi/2$. Thus there is at least one value ϕ_1 of ϕ for which the signed distance is zero and for this value the projections of A and B orthogonal to $L(\phi_1)$ meet at a point. Thus the line through that point perpendicular to $L(\phi_1)$ is a separating double support line and its upward normal vector is parallel to $L(\phi_1)$ which makes a clockwise angle ϕ_1 with the z-axis where $0 < \phi_1 \leq \pi/2$.

By a similar argument there is at least one separating double support line whose upward normal vector makes counterclockwise angle ϕ_2 with the z-axis where $0 < \phi_2 \leq \pi/2$.

To show that at least two of the lines whose existence was shown above are distinct consider the following two cases.

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Case I. The separation function at $(A, B, \pi/2)$ is negative.

Then the two lines whose existence was shown above are distinct.

Case II. The separation function at $(A, B, \pi/2)$ is 0.

Then the z-axis is a separating double support line. Thus the x-coordinates of the interior points of A all have the same sign. If the interior points of A all have negative x-coordinates then the image of the reflection of A about the z-axis will have positive x-coordinates. Thus without loss of generality one can assume that the interior of points of A have positive x-coordinates and that the interior points of B have negative x-coordinates. Thus the interior of A is contained in Quadrant IV and the interior of B is contained in Quadrant II.

Now let $\phi_1 = \pi/2$. Then the coordinates of A_{0f} may be labeled (x_2, z_1) and $(0, z_2)$ and the coordinates of B_{0f} may be labeled $(0, z_3)$ and (x_1, z_4) where $x_1 < 0 < x_2$ and $z_2 \leq 0 < c \leq z_3$. Let $\text{Proj}(x, z)(\phi)$ denote the orthogonal projection of the point (x, z) onto a clockwise rotation of the z-axis by angle ϕ.

Let $g_1(\phi) = \text{Proj}(0, z_i)(\phi) - \text{Proj}(x_1, z_4)(\phi) = z_i \cos\phi - x_1 \sin\phi - z_4 \cos\phi$ for $i = 2, 3$. Then g_1 is a continuous function of ϕ since it is a linear combination of continuous functions. Furthermore $g_1(\pi/2) = -x_1 > 0$. Thus there is a $\delta > 0$ such that $|\phi - \pi/2| < \delta$ implies $g_1(\pi/2) > 0$. Thus for $|\phi - \pi/2| < \delta$,

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$\text{Proj}(0, z_i)(\phi) - \text{Proj}(x_1, z_4)(\phi) > 0$ which implies $\text{Proj}(x_1, z_4)(\phi) < \text{Proj}(0, z_i)(\phi)$ for $i = 2, 3$.

Now let $g_2(\phi) = \text{Proj}(x_2, z_1)(\phi) - \text{Proj}(0, z_i)(\phi) = x_2 \sin\phi + z_1 \cos\phi - z_i \cos\phi$ for $i = 2, 3$. Then g_2 is a continuous function of ϕ since it is a linear combination of continuous functions. Furthermore $g_2(\pi/2) = x_2 > 0$. Thus there is a $\delta > 0$ such that $|\phi - \pi/2| < \delta$ implies $g_2(\pi/2) > 0$. Thus for $|\phi - \pi/2| < \delta$, $\text{Proj}(x_2, z_1)(\phi) - \text{Proj}(0, z_i)(\phi) > 0$ which implies $\text{Proj}(0, z_i)(\phi) < \text{Proj}(x_2, z_1)(\phi)$ for $i = 2, 3$.

Combining with the previous result then yields $\text{Proj}(x_1, z_4)(\phi) < \text{Proj}(0, z_i)(\phi) < \text{Proj}(x_2, z_1)(\phi)$ for $i = 2, 3$. Therefore for $\pi/2 - \delta < \phi < \pi/2$, $f_{0I}(\phi) = (z_2 - z_3)\cos\phi$. Thus for $\pi/2 - \delta < \phi < \pi/2$, $f(\phi) \leq f_{0I}(\phi) = (z_2 - z_3)\cos(\phi) < 0$. Thus $f(\pi/2 - \delta) < 0 < c = f(0)$. Therefore since f is continuous by Theorem 4.4.10, then by the intermediate value theorem $f(\phi) = 0$ for some value ϕ_1 of ϕ between 0 and $\pi/2 - \delta$. Thus $L(\phi_1)$ is parallel to a second distinct upward normal vector to a separating double support line.

The two cases are exhaustive. □

Lemma 4.6.2 *Let A and B be disjoint compact convex bodies in R^2 with positive area. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that* 0 *and* $c > 0$ *are endpoints of the*

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above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Suppose there are at least two distinct separating double support lines L_1 and L_2. Then L_1 and L_2 meet in a point.

Proof. Let L_1 and L_2 be two separating double support lines for A and B. Project R^2 onto a line orthogonal to L_1. Then A and B project to line segments and L_1 projects to the shared endpoint p of the two line segments and separates the interiors of the two line segments. Since L_2 is distinct from L_1 its projection will not be the single point p. Furthermore since L_2 supports both A and B its projection must include points from both line segments. Thus the projection of L_2 is not a single point. Thus L_2 is not parallel to L_1. Thus L_1 and L_2 meet in a point. □

Theorem 4.6.3 *Let A and B be disjoint compact convex bodies in R^2 with positive area. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xz coordinate system with positive orientation. Then there are exactly two separating double support lines at least one of which has an upward normal vector making an angle ϕ clockwise from the z-axis such that $0 < \phi \leq \pi/2$ and at least one of which has an upward normal vector making an angle ϕ counterclockwise from the z-axis such that $0 < \phi \leq \pi/2$.*

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Proof. Existence was proven above in Lemma 4.6.1. Suppose there are three separating double support lines L_1, L_2, and L_3. For $i = 1, 2, 3$ let H_{Ai} denote the closed half-plane determined by L_i and containing A. By Lemma 4.6.2 above L_1 and L_2 intersect in a point p.

Case I. $p \notin L_3$.

Then p is in the interior of one of the two half-planes determined by L_3. Without loss of generality assume that p is in the interior of H_{A3}. Let $a_1 \in L_1 \cap A$ and let $a_2 \in L_2 \cap A$. Then the rays pa_1 and pa_2 form the boundary of $H_{A1} \cap H_{A2}$. Thus A is contained in $H_{A1} \cap H_{A2}$ which is contained in the interior of H_{A3}. Therefore L_3 does not intersect A. Therefore L_3 is not a support line for A. This is a contradiction. Thus Case I cannot occur.

Case II. $p \in L_3$.

Since A and B are disjoint p is not in both. Without loss of generality assume $p \notin A$. Let $a_1 \in L_1 \cap A$ and let $a_2 \in L_2 \cap A$. Then the rays pa_1 and pa_2 form the boundary of $H_{A1} \cap H_{A2}$. Thus A is contained in $H_{A1} \cap H_{A2} \backslash p$ which is contained in the interior of H_{A3}. Therefore L_3 does not intersect A. Therefore L_3 is not a support line for A. This is a contradiction. Thus Case II cannot occur.

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Thus Cases I and II are exhaustive and neither case can occur. Thus there are exactly two separating double support lines. This proves the theorem. □

Corollary 4.6.4 *Let A and B be disjoint compact convex bodies in R^2. Then the points of separating double support on A partition the rest of the boundary of A into two disjoint continuous parts depending on whether the separation function is positive or negative. The closure of the portion of the boundary of A where the separation function is positive is the cap of A. An analogous statement may be made regarding the cap of B.*

Remark 4.6.5 *The statements of the theorems of this section and the next are made a bit cumbersome by the fact that it is not necessarily true that for rotation angle ϕ between zero and $\pi/2$ there is exactly one separating double support line corresponding to a clockwise rotation and exactly one separating double support line corresponding to a counterclockwise rotation. The difficulty is that the line through a shortest line segment between the two bodies may also be a separating double support line. In that case the normal vector to this support line would make an angle of exactly $\pi/2$ with the z-axis in either the clockwise or counter clockwise direction and there would be exactly one additional separating double support line. For a specific example see Figure 2.1 of Chapter 2.*

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4.7 Separating Double Support Planes in R^3

Theorem 4.7.1 *Let A and B be disjoint compact convex bodies in R^3. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xyz coordinate system with positive orientation. Parametrize almost all planes in R^3 by angle θ of rotation about the z-axis of the projection of the upward normal vector onto the xy-plane where $0 \leq \theta < 2\pi$, angle ϕ of the normal vector with the z-axis where $0 \leq \phi < \pi/2$, and by the intersection z of the plane with the z-axis where $-\infty < z < \infty$. Extend the (θ, ϕ) coordinates but not the z coordinates nonuniquely to planes parallel to the z-axis. Thus the range of the extended ϕ-coordinates is $0 \leq \phi \leq \pi/2$. Then there is at least one and at most two separating double support planes for each angle θ. Furthermore, if there are two such planes for a given angle θ then one of the planes has ϕ-coordinate $\pi/2$ and there are no additional such planes for angle $(\theta + \pi)$ mod 2π.*

Proof: Fix θ. Let $\alpha = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$. Project R^3 onto the plane containing α (for all values of ϕ) and the z-axis. The projections of A and B are convex bodies in R^2. The separating double support planes which are normal to α project to separating double support lines and vice versa. Thus by Theorem 4.6.3 there is at least one and at most two separating double

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support planes for each angle θ. Furthermore, if there are two such planes for a given angle θ then one of the planes has ϕ-coordinate $\pi/2$ and there are no additional such planes for angle $(\theta + \pi) \bmod 2\pi$. □

Corollary 4.7.2 *There is a one-to-one correspondence between separating double support planes and the angle θ for $0 \leq \theta < 2\pi$.*

In later chapters it will be useful to extend the notion of separation function to pairs of convex bodies in R^3.

Definition 4.7.3 *Let A and B be disjoint compact convex bodies in R^3. Let the z-axis be the line through a shortest line segment connecting A and B. Put a scale on the z-axis so that 0 and $c > 0$ are endpoints of the above line segment. Extend the z-axis to an xyz coordinate system with positive orientation. Let $L(\theta, \phi)$ be a rotation of the z-axis by angle ϕ about the y-axis followed by a rotation by angle θ about the z-axis. Define the* **separation function** $f = f(A, B, \theta, \phi)$ *as the signed distance between the orthogonal projections of A and B onto $L(\theta, \phi)$.*

Theorem 4.7.4 *Let A and B be disjoint compact convex bodies in R^3. Then for fixed (A, B) the separation function $f = f(A, B, \theta, \phi)$ is a continuous function of (θ, ϕ).*

Proof. The proof is essentially a repeat of the proofs of Section 4.4. For fixed (θ_0, ϕ_0) let A_{0C} denote the rectangle circumscribing A and let B_{0C} denote

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the rectangle circumscribing B with a set of sides orthogonal to the direction (θ_0, ϕ_0). Also let A_{0I} denote an inscribed line segment connecting support points of the support planes to A orthogonal to (θ_0, ϕ_0) and let B_{0I} denote an inscribed line segment connecting support points of the support planes to B orthogonal (θ_0, ϕ_0). Let f denote the separation function of A and B, let f_{0C} denote the separation function of A_{0C} and B_{0C}, and let f_{0I} denote the separation function of A_{0I} and B_{0I}. We then use a delta-epsilon argument and the fact that f is bounded by f_{0C} and f_{0I} to show that f is continuous. □

4.8 Examples and Counter Examples

The assumption of compactness was sufficient and convenient but not absolutely necessary for the theorems of this chapter. Consider for example the parabolas

$$s_A(x) = (x, -x^2) \quad s_B(x) = (-x, x^2 + c)$$

where $c > 0$. We differentiate to get the tangent vectors

$$s_{Ax}(x) = (1, -2x) \quad s_{Bx}(x) = (-1, 2x).$$

From these we compute the upward unit normal vectors

$$n_A(x_A) = \frac{(2x_A, 1)}{\sqrt{4x_A^2 + 1}}$$

$$n_B(x_B) = \frac{(2x_B, 1)}{\sqrt{4x_B^2 + 1}}.$$

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At separating double tangent points the upward unit normal vectors must be the same. Thus $x_A = x_B$ and we can drop the subscripts. Then solving

$$n_A(x) \cdot s_A(x) = n_B(x) \cdot s_B(x)$$

yields

$$x = \pm\sqrt{\frac{c}{2}}.$$

These two values of x correspond to the two separating double support lines. The parabolas are not compact. Thus compactness is not a necessary condition for the existence of a pair of separating double support lines. See Figure 4.1 for a graph of the envelope of separating double support lines for $c = 9/2$

A second set of examples shows that although the compactness condition may be relaxed it cannot be dropped entirely without substituting some other sufficient condition. Consider the one-sheeted hyperbolas

$$s_A(x) = (x, -\sqrt{x^2+1}) \quad s_B(x) = (-x, \sqrt{x^2+1}+c)$$

where c is a real number. We differentiate to get the tangent vectors

$$s_{Ax}(x) = \left(1, -\frac{x}{\sqrt{x^2+1}}\right) \quad s_{Bx}(x) = \left(-1, \frac{x}{\sqrt{x^2+1}}\right).$$

From these we compute the upward unit normal vectors

$$n_A(x_A) = \frac{(x_A, \sqrt{x_A^2+1})}{\sqrt{2x_A^2+1}}$$

$$n_B(x_B) = \frac{(x_B, \sqrt{x_B^2+1})}{\sqrt{2x_B^2+1}}.$$

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At separating double tangent points the upward unit normal vectors must be the same. Thus $x_A = x_B$ and we can drop the subscripts on x. Then solving

$$n_A(x) \cdot s_A(x) = n_B(x) \cdot s_B(x)$$

yields

$$\sqrt{x^2+1} = -\frac{2}{c}$$

which implies that in contrast to the previous examples there can only be separating double support lines if $c < 0$. Solving for x then implies

$$x = \pm\frac{\sqrt{4-c^2}}{c}.$$

These two values of x correspond to the two separating double support lines which exist if $c < 0$.

Figure 4.2 shows a pair of one-sheeted hyperbolas when $c = 1$. The asymptotes are also shown. The reason that there is no separating double support line is that the slope of the hyperbolas is never steeper than the ± 1 slope of the asymptotes and the figures are too far apart to share tangent lines with such moderate slopes. Thus we can only drop the compactness condition if we replace it with some other condition.

Figure 4.3 for a graph of the envelope of separating double support lines for a pair of one-sheeted hyperbolas when $c = -3/2$. Once again the slopes of each sheet are between -1 and 1 but now the two sheets are close enough that they can share tangent lines with moderate slopes.

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Thus we may drop the compactness assumption only if we replace it with some other condition which may be more complicated. Thus it is often convenient to assume compactness.

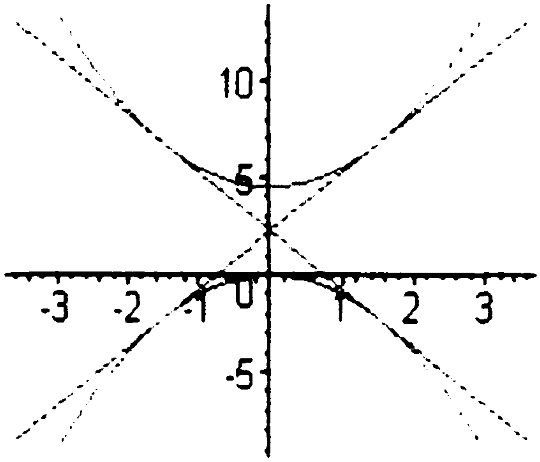

Figure 4.1: Support Lines of a Pair of Parabolas

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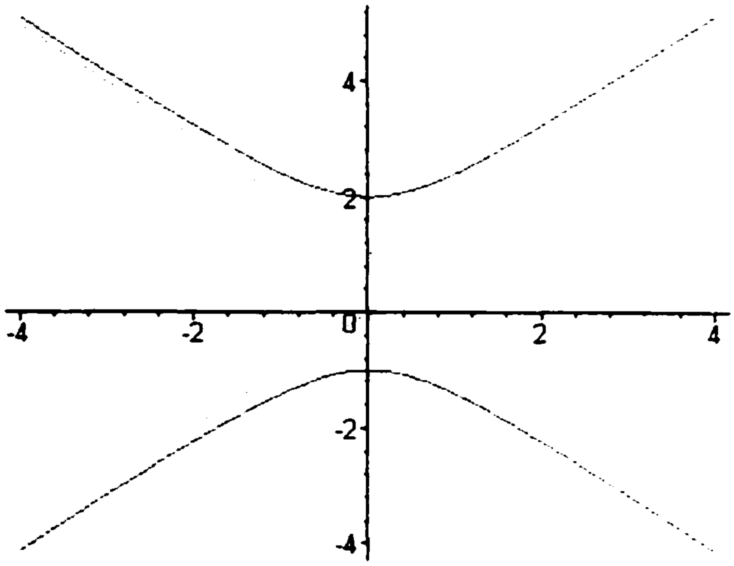

Figure 4.2: Pair of One-Sheeted Hyperbolas With Asymptotes

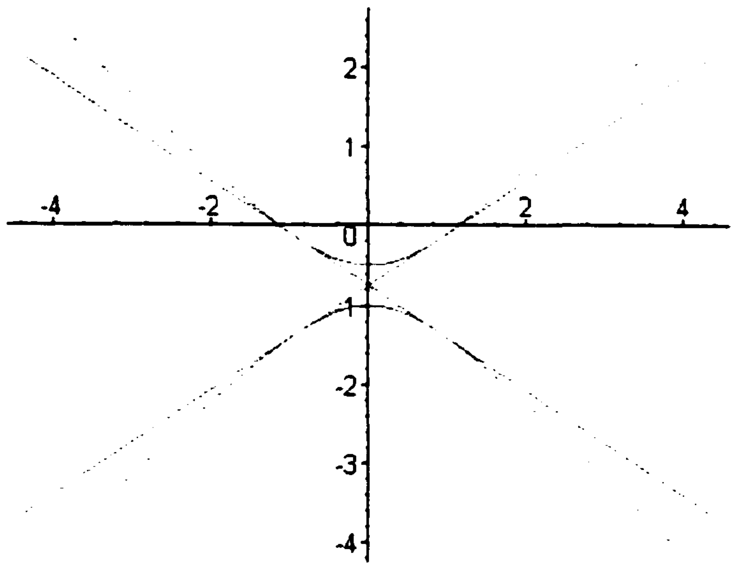

Figure 4.3: Support Lines of a Pair of One-Sheeted Hyperbolas

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CHAPTER 5

MEASURE OF PLANES SEPARATING TWO CONVEX POLYHEDRA IN R^3

5.1 Introduction

A classical formula for the measure of planes intersecting a compact convex polyhedron in R^3 is

$$\frac{1}{2}\sum |\nu||V|$$

where the sum is taken over all edges ν and $|\nu|$ is the length of the edge and

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$|V|$ is the exterior angle of the adjoining faces at the edge. See for example Ambartzumian (1990, 114).

We seek to use the results of Chapters 3 and 4 to derive an analogous formula for the measure of planes separating two disjoint compact convex polyhedra in R^3. We assume that each polyhedron is nondegenerate (i.e. not all of the vertices of the polyhedron are coplanar.)

5.2 Wedge Coefficients

Theorem 5.2.1 *Let P be the set of vertices of a pair of nondegenerate disjoint compact convex polyhedra in R^3. Then the measure of the set of planes separating the polyhedra is the sum over all allowable separating wedges of the wedge functions taken with a minus sign if the needle of the wedge is an edge of one of the polyhedra and taken with a plus sign otherwise.*

Proof. Let Q be the set of planes which separate the two polyhedra. Note that Q is also the set of planes which separate the two sets of vertices. Thus Theorem 3.2.2 may be applied.

To compute the coefficients $c_w(Q)$ consider four cases below. In each case let A and B denote the two polyhedra. Let H^+ denote the half-space which is determined by a plane of the wedge and contains the interior of B if such a

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half-space exists. Otherwise arbitrarily denote one of the half-spaces as H^+. Let H^- denote the other half-space determined by the plane of the wedge.

Case I.

If a plane of a wedge intersects the interior of one or more of the polyhedra then disturbing it slightly will not alter that condition and therefore the plane will not separate the two polyhedra. Thus all four $I_Q(i,j) = 0$ and thus $c_s(Q) = 0$.

Case II.

Suppose that the interiors of the two polyhedra are in the same half-space H^+ determined by a plane of the wedge. Then all of the vertices of the two polyhedra except the vertices of the wedge are in H^+. Since by assumption each polyhedron is nondegenerate then at least 1 vertex from each polyhedron is in H^+. If the plane of the wedge is perturbed in such a way that both vertices of the wedge are in $H^{+\prime}$ then all of the vertices of both polyhedra will be in $H^{+\prime}$ and the perturbed plane will not separate the polyhedra. If the plane of the wedge is disturbed in such a way that at least one of the vertices of the wedge is in the other half-plane $H^{-\prime}$ then the plane separates that vertex from the other vertices of the same polyhedron and thus does not separate the two polyhedra. Thus under either scenario all four $I_Q(i,j) = 0$ and therefore $c_s(Q) = 0$.

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Case III.

Suppose a plane of a wedge separates the two polyhedra and that the needle of the wedge is an edge of a polyhedron. Let A be the polyhedron for which the needle is an edge and let B be the other polyhedron. Then the half-space H^- contains all of the vertices of A except the vertices of the wedge. Thus if the plane of the wedge is perturbed in such a way that both vertices of the needle of the wedge are in $H^{-\prime}$ then all of the vertices of A will be in a different half-space than the vertices of K_2 and thus the perturbed plane will separate the two polyhedra. If the plane of the wedge is perturbed in such a way that at least one of the vertices of the needle of the wedge is in $H^{+\prime}$ then the perturbed plane will separate the vertices of A and thus will not separate the two polyhedra. Thus the coefficient $I_Q(\bar{i}, \bar{j})$ is 1 and the remaining three coefficients $I_Q(i, j.)$ are 0 and therefore $c_s(Q) = -1$. If B is the polyhedron for which the needle is an edge then by a similar argument the coefficient $I_Q(\overset{+}{i}, \overset{+}{j})$ is 1 and the remaining three coefficients $I_Q(i, j)$ are 0 and therefore in either case $c_s(Q) = -1$.

Case IV.

Suppose a plane of a wedge separates the two polyhedra and that the needle of the wedge has endpoints on different polyhedra. Let a and b denote the

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vertices of the needle which are vertices of polyhedra A and B respectively. If the plane of the wedge is perturbed in such a way that $H^{+\prime}$ contains b and $H^{-\prime}$ contains a then $H^{+\prime}$ will contain all of the vertices of B and $H^{-\prime}$ will contain all of the vertices of A and the perturbed plane will separate the polyhedron pair. Otherwise the perturbed plane will separate the vertices of one of the polyhedra and will not separate the polyhedron pair. Thus one of the two coefficients $I_Q(\overset{+}{i}, \overset{-}{j})$ and $I_Q(\overset{-}{i}, \overset{+}{j})$ is 1 and the other three coefficients $I_Q(i, j)$ are 0 and thus $c_s(Q) = 1$.

The four cases are exhaustive. □

5.3 Neighboring Wedges

Theorem 5.3.1 *Let A and B be disjoint nondegenerate compact convex polyhedra in R^3. Then each face of an allowable separating wedge whose needle contains a vertex of each polyhedron contains exactly one needle of another allowable separating wedge such that this needle also contains a vertex of each polyhedron.*

Proof. Consider one of the two faces of a separating wedge whose needle contains vertices from each polyhedron. It follows from the definition of a

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wedge that this wedge face contains at least one additional vertex of A or B in addition to the endpoints of the needle of the wedge.

Case I.

All of the vertices on this wedge face other than endpoints of the original needle are from the same polyhedron. Without loss of generality suppose all of these vertices are vertices of A.

Let a_1 be a vertex of A and let b_1 be a vertex of B and let a_1 and b_1 be endpoints of the original needle. Form the convex hull A' of the vertices of A which are on the face of the wedge. Since A is convex then the vertices of A' coincide with the vertices of A which are on the face of the wedge. Now use b_1 and A' to form a cone. Since a_1b_1 is the needle of a separating wedge then the line forming one of the boundaries of the cone will go through a_1b_1. The other line will go through b_1 and a second vertex of A', call it a_2. Note that the line through a_2 and b_1 divides the wedge face into two half-planes and that all of the vertices of A' are in the same closed half-plane.

Project R^3 onto a plane P orthogonal to the line through the vertices a_2 and b_1. Then the projection of the face of the wedge is a separating double support line for the polygons Proj$A$ and Proj$B$. Also the projections of the vertices of A' will lie in the same half-line determined by the point Proja_2 = Projb_1. Since no other vertices are on the line, the line may be rotated slightly

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and still separate the interiors of the two polygons. This rotation corresponds to a rotation of the face of the wedge about the line through a_2b_1. Since the plane is an orthogonal projection onto the line then the rotated plane will still separate the polyhedra. Thus a_2 and b_1 are endpoints of a needle of another separating wedge.

No other vertices of A' may be paired with b_1 to form the needle of a separating wedge because such a needle would separate vertices of A'.

Case II.

Each polyhedron has at least two vertices on this wedge face and one of the polyhedra has exactly two vertices on the wedge face such that one of the vertices is contained in the convex hull of all of the vertices of the wedge face. Without loss of generality assume that the polyhedron B has vertices b_1 and b_2 and no other vertices on the wedge face and that b_1 is interior to the convex hull of all of the other vertices of the wedge face.

Note that b_2 must be a vertex of the convex hull of all of the other vertices of the wedge face. Otherwise all points of B on the wedge face are interior points of the convex hull and thus any line through an interior point will separate points of A and thus any plane through such a line cannot be rotated about that line without intersecting the interior of A.

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Note also that an edge of the above convex hull cannot be the needle of a separating wedge because points of A and B would be in the same half-plane determined by the line through the edge. Thus the wedge face could not be rotated about that line without intersecting the interiors of either A or B.

Now as in Case I let A' be the convex hull of the vertices of A which are on the face of the wedge and use b_1 and A' to form a cone. Let a_1 and a_2 be the vertices of A' that are on the boundary of the cone. Then the only possible needles of separating wedges with needles on different polyhedra are a_1b_1 and a_2b_1. Without loss of generality assume that the original needle was a_1b_1. Let B' be the line segment b_1b_2.

Note that the line through a_2b_1 separates points of A' and B'. As in Case I above project R^3 onto a plane P orthogonal to the line through the vertices a_2 and b_1. Then the projection of the face of the wedge is a separating double support line for the polygons Proj$A$ and Proj$B$. Also the projections of the vertices of A' and B' will lie in different half-lines determined by the point Proja_2 = Projb_1. Thus the line may be rotated slightly and still separate the interiors of the two polygons. This rotation corresponds to a rotation of the face of the wedge about the line through a_2b_1. Since the plane is an orthogonal projection onto the line then the rotated plane will still separate the polyhedra. Thus a_2 and b_1 are endpoints of a needle of another separating wedge.

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Case III.

Each polyhedron has at least two vertices which are vertices of the convex hull of all of the vertices which are on the wedge face.

As in Cases I and II above let A' denote the convex hull of the vertices of A which are on the wedge face and let B' denote the convex hull of the vertices of B which are on the wedge face. Then from Crofton's Theorem there are exactly two separating double support lines to A' and B'. Without loss of generality assume that one of the lines goes through a_1 and b_1 and that the other line goes through a_2 and b_2 where $a_1, a_2 \in A'$ and $b_1, b_2 \in B'$. Thus using arguments similar to those in Cases I and II above the only needles of separating wedges with vertices of different polyhedra on this wedge face are a_1b_1 and a_2b_2. Without loss of generality take a_1b_1 to be the original needle and take a_2b_2 to be the additional needle. Crofton's Theorem also implies that these two needles meet in a point interior to each of the needles.

Cases I. II. and III. are exhaustive. □

5.4 Wedge Cycle

Theorem 5.4.1 *Let A and B be disjoint compact convex polyhedra in R^3. Suppose that A and B have exactly n separating wedges whose needles contain*

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vertices from different polyhedra. Let the z-axis be the line through the shortest line segment connecting A and B. Put a scale the z-axis by letting the intersection with A be 0 and the intersection with B be $c > 0$. Extend the z-axis to an xyz-coordinate system with positive orientation. Let P be a plane of a separating wedge whose needle contains vertices of different polyhedra. Let θ be the angle with the x-axis of the projection of the normal vector of P onto the xy-plane. Then:

(i) The values of θ on each wedge form an interval modulo 2π.

(ii) The wedges may be ordered according to the values of θ and the values of θ on wedges $w_1, w_2, \cdots, w_n$ are respectively all of the values of θ in the intervals $[\theta_1, \theta_2], [\theta_2, \theta_3], \cdots, [\theta_n, \theta_1]$.

(iii) One face of each wedge contains the needle of the next consecutive wedge. The other face of each wedge contains the needle of the preceding wedge.

Proof. Let ψ be the angle of rotation about the needle of the normal vector of a plane P of a separating double support wedge with vertices on different polyhedra. Then the angle θ may be obtained from ψ by a composition of rotations, computing the normal vector, projections, dot product, and trig

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functions. Thus θ is a composition of continuous functions and therefore is a continuous function of ψ. Furthermore by Theorem 4.7.1 there is only one support plane for each value of θ. Thus θ is monotonic. Thus the extreme values of θ for each wedge occur at the faces. This proves (i).

Then Theorems 5.3.1, 4.7.1, and part (i) above together imply that the value of θ on a face is a maximum for one wedge and a minimum for the other wedge. This proves (ii) and (iii). □

5.5 Envelope and Caps

Definition 5.5.1 *Define the faces of the* **canonical envelope of separating double support planes** *for two disjoint nondegenerate convex polyhedra (or just* **canonical envelope** *for brevity) as follows:*

(i) If the convex hull of the two needles of a face of a wedge is a triangle then a face of the envelope is that triangle.

(ii) If the convex hull of the two needles of a face of a wedge is a quadrilateral then the needles divide the quadrilateral into four triangles. The face of the envelope is the union of the two triangles which are each bounded by an edge of a polyhedron.

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The canonical envelope is then the union of the above faces over wedge needles with a vertex on each polyhedron.

Theorem 5.5.2 *The canonical envelope is an envelope. Furthermore the separating wedges of the envelope of separating double support planes for two disjoint nondegenerate compact convex polyhedra are the same as the separating double support wedges for the pair of polyhedra.*

Proof. This is true by construction.

Theorem 5.5.3 *The measure of planes separating two disjoint nondegenerate compact convex polyhedra in R^3 is equal to the wedge function over the separating wedges of the envelope minus the wedge function over the separating wedges of the caps.*

Proof. By Theorem 5.2 above the measure of planes separating the two polyhedra is the sum over all allowable separating wedges of the wedge functions taken with a minus sign if the needle of the wedge is an edge of one of the polyhedra and taken with a plus sign otherwise.

Since A is convex there is exactly one support plane to A for each direction (θ, ϕ) of the outward normal vector. Thus the values of (θ, ϕ) for the planes of an outer wedge of A are distinct from the values of (θ, ϕ) for the planes of

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other outer wedges of A. The planes of each outer wedge of A are rotations of each other through an angle. Thus the image of the unit normal vector of these planes is a geodesic curve on the unit sphere. The corresponding geodesic curves of outer wedges which share a face will meet at a point on the unit sphere corresponding to the unit outward normal vector of the plane of the face. Thus the values of (θ, ϕ) for planes of outer wedges of A are distinct and pathwise connected.

We may evaluate the separation function of A and B on the normal directions of the planes of the outer wedges of edges of A. Since by Theorem 4.7.4 the separation function is a continuous function of (θ, ϕ), the edges of A for which the separation function is zero bound the edges of A for which the separation function is positive. These two sets of edges may or may not be the same. We note that planes for which the separation function is positive are separating planes and that the value of the separation function on the support plane through the closest point of A to B is positive. Furthermore the edges of A for which the separation function is zero are points of separating double support and thus the edges of A for which the separation function is positive are edges of the cap of A. Thus the set of separating wedges on the cap of A is precisely the set of separating wedges on the edges of A. Likewise the set of separating wedges on the cap of B is precisely the set of separating wedges on the edges of B.

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Furthermore the separating wedges whose needle contains a vertex of each polyhedron are the wedges of the envelope by construction. This proves the theorem. □

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CHAPTER 6

A POLYHEDRAL EXAMPLE: MEASURE OF PLANES SEPARATING CUBES

6.1 Introduction

In this chapter we will specialize the formulas for the measure of planes separating polyhedra to the measure of planes separating cubes.

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6.2 Planes Separating Parallel Coaxial Cubes

Let $(x, y, z) \in R^3$. Consider two parallel cubes with sides of length s and distance c apart and a common axis of symmetry. Without loss of generality we can coordinatize the vertices as follows: Cube A has facing vertices

$$\left(\frac{s}{2}, \frac{s}{2}, 0\right), \left(\frac{s}{2}, -\frac{s}{2}, 0\right), \left(-\frac{s}{2}, -\frac{s}{2}, 0\right), \left(-\frac{s}{2}, \frac{s}{2}, 0\right)$$

and Cube B has facing vertices

$$\left(\frac{s}{2}, \frac{s}{2}, c\right), \left(\frac{s}{2}, -\frac{s}{2}, c\right), \left(-\frac{s}{2}, -\frac{s}{2}, c\right), \left(-\frac{s}{2}, \frac{s}{2}, c\right).$$

Because of the symmetry of this pair of cubes all of the tangent planes will meet at the central point

$$\left(0, 0, \frac{c}{2}\right).$$

Thus the envelope of separating double support planes can be formed by joining the facing edges of the two cubes to the central point. The envelope is shown in Figure 6.3. A straightforward computation gives the total measure of the wedges over the four edges of the envelope as

$$2\sqrt{2s^2 + c^2}\cos^{-1}\left(\frac{s^2}{c^2 + s^2}\right)$$

and the total measure of the wedges over the eight edges of the caps as

$$4s\cos^{-1}\left(\frac{s}{\sqrt{c^2 + s^2}}\right).$$

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Thus the measure of planes separating the two cubes is

$$2\sqrt{2s^2+c^2}\cos^{-1}\left(\frac{s^2}{c^2+s^2}\right)-4s\cos^{-1}\left(\frac{s}{\sqrt{c^2+s^2}}\right).$$

6.3 Planes Separating Coaxial Cubes Rotated by $\pi/4$

Let $(x, y, z) \in R^3$. Consider two cubes with sides of length s and distance c apart such that one cube is rotated by an angle of $\pi/4$ about a common axis of symmetry. Without loss of generality we can coordinatize the vertices as follows: Cube A has facing vertices

$$\left(\frac{s}{2},\frac{s}{2},0\right),\left(\frac{s}{2},-\frac{s}{2},0\right),\left(-\frac{s}{2},-\frac{s}{2},0\right),\left(-\frac{s}{2},\frac{s}{2},0\right)$$

and Cube B has facing vertices

$$\left(0,\frac{\sqrt{2}s}{2},c\right),\left(-\frac{\sqrt{2}s}{2},0,c\right),\left(0,-\frac{\sqrt{2}s}{2},c\right),\left(\frac{\sqrt{2}s}{2},\frac{s}{2},c\right).$$

The envelope is formed by pasting together triangular flats which are formed from the convex hull of the points of contact of the separating double support planes. Thus the faces of the envelope will be triangular flats consisting of the convex hull of an edge of a facing face and a remote facing vertex from the other cube. By inspection we see that the vertices

$$\left(\frac{s}{2},-\frac{s}{2},0\right),\left(\frac{s}{2},\frac{s}{2},0\right),\left(-\frac{s}{\sqrt{2}},0,c\right) \tag{6.3.1}$$

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lie on a separating double support plane and form a face of the lower envelope. Subtracting the third vertex from each of the first two vertices yields two support vectors

$$\left(\frac{s}{2}(1+\sqrt{2}), -\frac{s}{2}, -c\right) \quad \text{and} \quad \left(\frac{s}{2}(1+\sqrt{2}), -\frac{s}{2}, -c\right).$$

Taking the cross product and normalizing yields the unit normal vector

$$\mathbf{n_A} = \frac{\left(2c, 0, s(1+\sqrt{2})\right)}{\sqrt{4c^2 + (3+2\sqrt{2})s^2}}$$

to this face of the envelope.

By changing one of the vertices we obtain a set of vertices

$$(6.3.2) \qquad \left(-\frac{s}{\sqrt{2}}, 0, c\right), \left(0, -\frac{s}{\sqrt{2}}, c\right), \left(\frac{s}{2}, \frac{s}{2}, 0\right)$$

of an adjoining face of the upper envelope. Again subtracting the third vertex from each of the other vertices yields the support vectors

$$\left(-\frac{s}{2}(1+\sqrt{2}), -\frac{s}{2}, c\right) \quad \text{and} \quad \left(-\frac{s}{2}, -\frac{s}{2}(1+\sqrt{2}), c\right).$$

Again taking the cross product and normalizing yields the unit normal vector

$$\mathbf{n_B} = \frac{\left(\sqrt{2}c, \sqrt{2}c, s(1+\sqrt{2})\right)}{\sqrt{4c^2 + (3+2\sqrt{2})s^2}}$$

to this adjoining face of the envelope. Thus the angle between these adjoining faces is

$$\cos^{-1}\left(\mathbf{n_A} \cdot \mathbf{n_B}\right) = \cos^{-1}\left(\frac{2\sqrt{2}c^2 + s^2(3+2\sqrt{2})}{4c^2 + s^2(3+s\sqrt{2})}\right).$$

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These faces share an edge with vertices

$$\left(-\frac{s}{\sqrt{2}}, 0, c\right) \quad \text{and} \left(\frac{s}{2}, \frac{s}{2}, 0\right).$$

The length of this edge is

$$\frac{1}{2}\sqrt{4c^2 + 2s^2(2 + \sqrt{2})}.$$

Thus the wedge function for these two adjoining faces is

$$\frac{1}{2} \cdot \text{length} \cdot \text{angle} = \frac{1}{4}\sqrt{4c^2 + 2s^2(2 + \sqrt{2})} \cos^{-1}\left(\frac{2\sqrt{2}c^2 + s^2(3 + 2\sqrt{2})}{4c^2 + s^2(3 + s\sqrt{2})}\right).$$

Because of the rotational symmetry of this pair of cubes, the wedge function is the same for the other 8 wedges of the envelope.

We next compute the wedge function on the caps. The angle between the envelope face and a facing face of the cube is

$$\cos^{-1}(\mathbf{n_A} \cdot (0, 0, 1)) = \cos^{-1}\left(\frac{s(1 + \sqrt{2})}{\sqrt{4c^2 + s^2(3 + 2\sqrt{2})}}\right).$$

The edge of this wedge is the edge of the cube which has length s. Thus the wedge function on this wedge is

$$\frac{1}{2} \cdot \text{length} \cdot \text{angle} = \frac{1}{2} s \cos^{-1}\left(\frac{s(1 + \sqrt{2})}{\sqrt{4c^2 + s^2(3 + 2\sqrt{2})}}\right).$$

Thus the measure of planes separating the cubes is the sum of the 8 wedges of the envelope minus the sum of the eight wedges of the cap which is

$$2\sqrt{4c^2 + 2s^2(2 + \sqrt{2})} \cos^{-1}\left(\frac{2\sqrt{2}c^2 + s^2(3 + 2\sqrt{2})}{4c^2 + s^2(3 + s\sqrt{2})}\right)$$

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$$-4s\cos^{-1}\left(\frac{s(1+\sqrt{2})}{\sqrt{4c^2+s^2(3+2\sqrt{2})}}\right).$$

See Figure 6.1 for a picture of the cubes and Figures 6.4 and 6.5 for a picture of the envelope.

6.4 Empirical Calculations on Cubes

A QBasic program was written which takes the position of the cubes as input and then uses a random number generator to randomly place the cube pair on a grid of random planes. These random placements are shown visually on the screen. The proportion of pairs which are separated by a plane is then computed and compared to the theoretical probability.

When one runs the program the menus and input and output information are displayed on a sequence of six screens. See Figures 6.8, 6.9, and 6.10 for typical computer input and output as displayed on the screen while running the programs. To save space here each figure contains the information shown on two screens. See Appendix B for the actual QBasic computer program.

The empirical evidence was not very convincing. The program was run several times for as many as $10,000$ pairs of cubes and the theoretical and empirical probabilities were not necessarily close. There are several possibilities as to why this might be so. It might be that convergence is very slow. It might

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be that our program is defective and for example somehow does not uniformly distribute the placements of the cube pairs on the grid.

The program was modeled after a similar QBasic program written by Temple faculty member Eric Grinberg which randomly placed line segments on a grid of parallel lines and used that to compute the probability that a randomly placed line segment would intersect a line of the grid.

6.5 Cube Figures

The mathematical computer program Maple V Release 6 was used to create graphs of the cubes and of the envelopes.

Figures 6.1 and 6.2 show pairs of cubes in various orientations with respect to each other.

Figure 6.3 shows the envelope of separating double support planes for a pair of cubes where the second cube is a translation of the first cube in a direction parallel to a face of the first cube. The envelope is a cone. In this case the separating double support planes all meet at the vertex of the cone.

Figure 6.4 shows the envelope of separating double support planes for a pair of cubes where the second cube is obtained from the first cube by first rotating by angle $\pi/4$ about an axis of symmetry of the first cube and then translating in a direction parallel to that axis of symmetry. Note that the

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envelope contains self-intersections. Figure 6.5 shows two adjoining faces of the envelope. Note that although non-adjoining faces may meet in interior points, adjoining faces meet only along an edge.

Figures 6.6 and 6.7 show the envelope of separating double support planes for a pair of cubes where the second cube is obtained from the first cube by first rotating by angle $\pi/4$ about an axis of symmetry of the first cube and then translating in a direction perpendicular to that axis of symmetry. For this example the nature of the envelope depends on how far the translation is. A straightforward calculation shows that if the translation is more than one and a half times the side then six vertices of the moved cube are points of separating double tangency and thus vertices of the envelope. If the translation is less than one and a half times the side then only two vertices of the moved cube are points of separating double tangency and vertices of the envelope.

As an example of the type of calculations used in graphing the envelope we show the computation of the envelope for the cubes of Section 6.3 below. These cubes appear in Figures 6.4 and 6.5.

We somewhat arbitrarily specialize Formula 6.3.1 for the vertices of a face of the lower envelope to the case where $c = 4$ and $s = 2$ to get

$$(1, -1, 0), (1, 1, 0), \text{ and} (-\sqrt{2}, 0, 4)$$

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We use these vertices to parametrize the face as

$$\mathbf{f}_{\mathbf{A1}}(p,q) = (1-q)\big[(1-p)(1,-1,0) + p(1,1,0)\big] + q(-\sqrt{2},0,4)$$

$$= \Big(1-q(1+\sqrt{2}),(1-q)(2p-1),4q\Big).$$

Taking advantage of the symmetry of the pair of cubes we repeatedly rotate the above face about the z-axis by angle $\pi/2$ to get the other faces of the lower envelope. Thus multiplying $\mathbf{f}_{\mathbf{A1}}$ by the rotation matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives the other four faces of the lower envelope as

$$\mathbf{f}_{\mathbf{A2}}(p,q) = \Big((1-q)(2p-1),-1+q(1+\sqrt{2}),4q\Big).$$

$$\mathbf{f}_{\mathbf{A3}}(p,q) = \Big(-1+q(1+\sqrt{2}),-(1-q)(2p-1),4q\Big).$$

$$\mathbf{f}_{\mathbf{A4}}(p,q) = \Big(-(1-q)(2p-1),1-q(1+\sqrt{2}),4q\Big).$$

Likewise we specialize Formula 6.3.2 for the vertices of a face of the upper envelope for $c = 4$ and $s = 2$ to get

$$(-\sqrt{2},0,4),(0,-\sqrt{2},4),\ \text{and}(1,1,0).$$

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As before we use these vertices to parametrize the face as

$$\mathbf{f_{B1}}(p,q) = (1-q)(1,1,0) + q\Big[(1-p)(-\sqrt{2},0,4) + p(0,-\sqrt{2},4)\Big]$$
$$= \Big(1-q+\sqrt{2}(p-1)q, 1-q-\sqrt{2}pq, 4q\Big).$$

Again taking advantage of the symmetry of the pair of cubes we repeatedly rotate the above face about the z-axis by angle $\pi/2$ to get the other faces of the lower envelope. Thus multiplying $\mathbf{f_{B1}}$ by the rotation matrix yields

$$\mathbf{f_{B2}}(p,q) == \Big(1-q-\sqrt{2}pq, -1+q-\sqrt{2}(p-1)q, 4q\Big),$$

$$\mathbf{f_{B3}}(p,q) == \Big(-1+q-\sqrt{2}(p-1)q, -1+q+\sqrt{2}pq, 4q\Big), \text{ and}$$

$$\mathbf{f_{B4}}(p,q) == \Big(-1+q+\sqrt{2}pq, 1-q+\sqrt{2}(p-1)q, 4q\Big).$$

The envelope was then graphed using the following Maple commands:

```
with(plots):
display([
 plot3d([1-q*(1+sqrt(2)),(1-q)*(2*p-1),4*q],
   p=0..1,q=0..1),
 plot3d([(1-q)*(2*p-1),-1+q*(1+sqrt(2)),4*q],
   p=0..1,q=0..1),
 plot3d([-1+q*(1+sqrt(2)),-(1-q)*(2*p-1),4*q],
   p=0..1,q=0..1),
 plot3d([-(1-q)*(2*p-1),1-q*(1+sqrt(2)),4*q],
   p=0..1,q=0..1),
 plot3d([1-q+sqrt(2)*(p-1)*q,1-q-sqrt(2)*p*q,4*q],
   p=0..1,q=0..1),
 plot3d([1-q-sqrt(2)*p*q,-1+q-sqrt(2)*(p-1)*q,4*q],
   p=0..1,q=0..1),
 plot3d([-1+q-sqrt(2)*(p-1)*q,-1+q+sqrt(2)*p*q,4*q],
   p=0..1,q=0..1),
 plot3d([-1+q+sqrt(2)*p*q,1-q+sqrt(2)*(p-1)*q,4*q],
   p=0..1,q=0..1)]);
```

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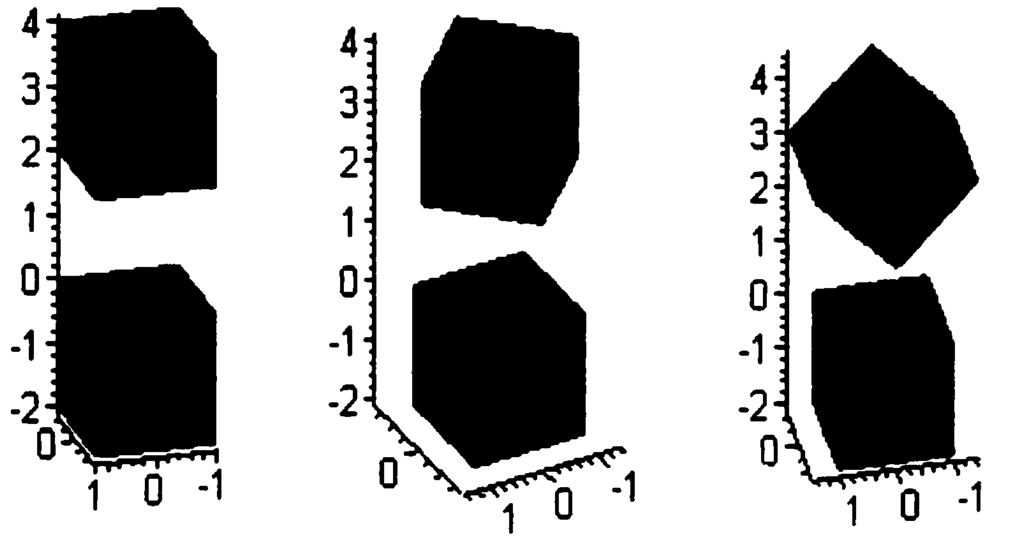

Figure 6.1: Cubes: Parallel. Rotated About z-axis. and Rotated About y-axis

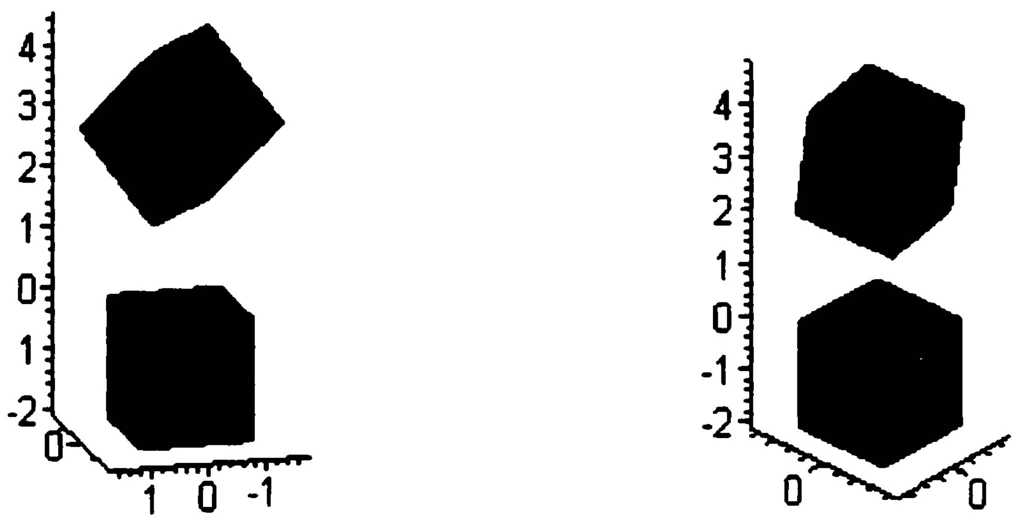

Figure 6.2: Doubly Rotated Cubes

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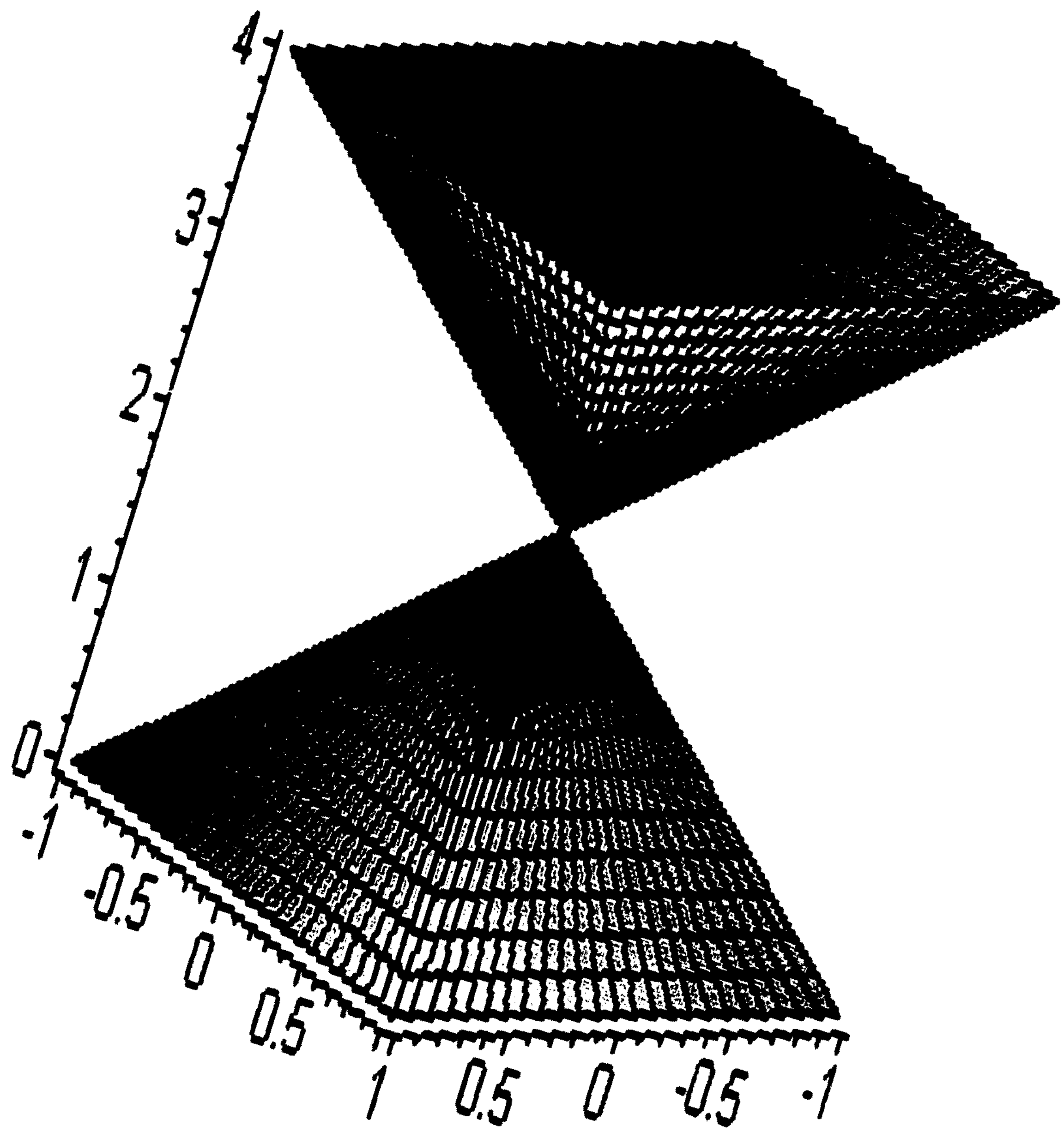

Figure 6.3: Parallel Cubes Envelope

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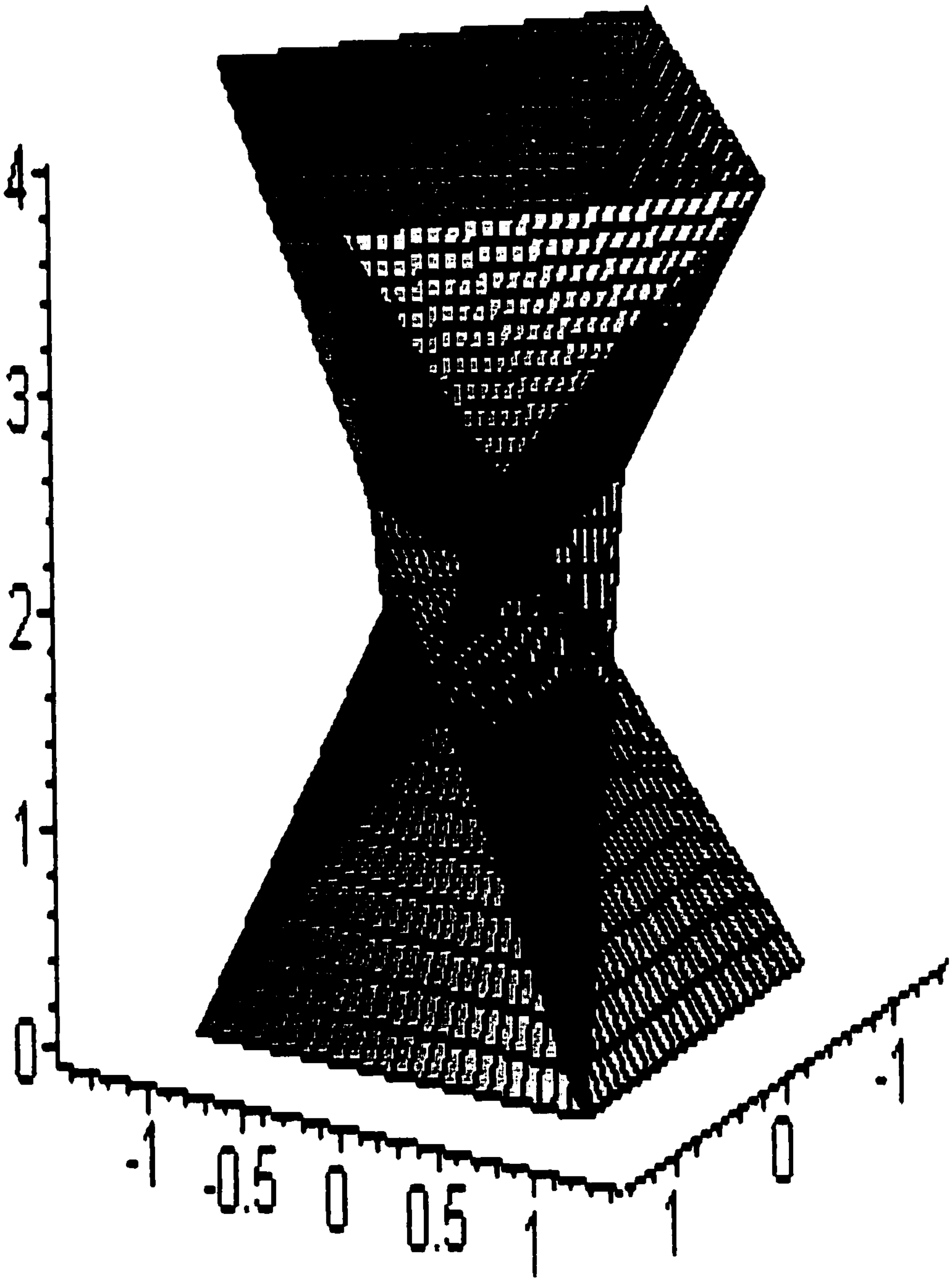

Figure 6.4: Envelope for Cubes Rotated an Angle of $\pi/4$ About z-axis

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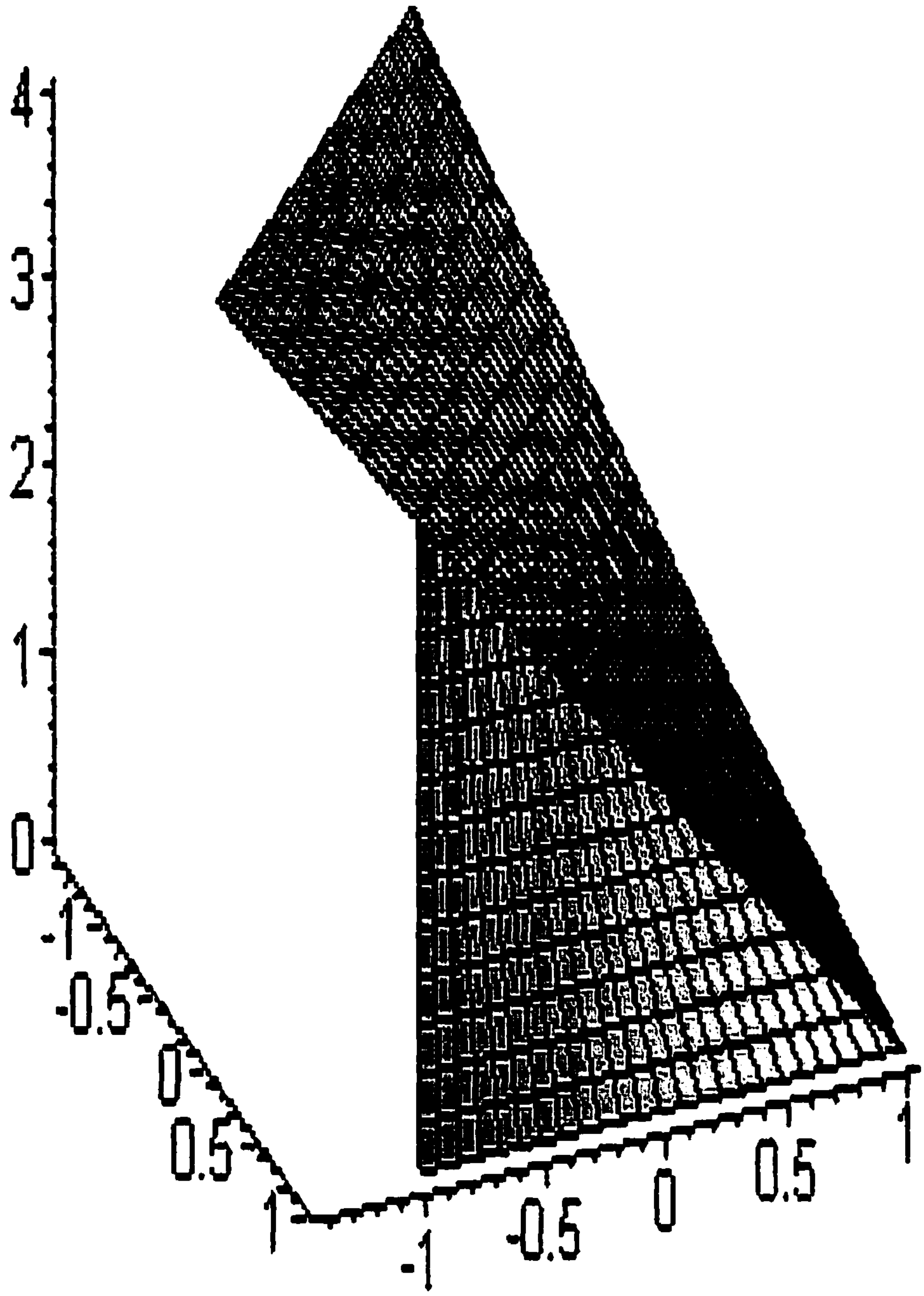

Figure 6.5: Consecutive Faces of Envelope for Rotated Cubes

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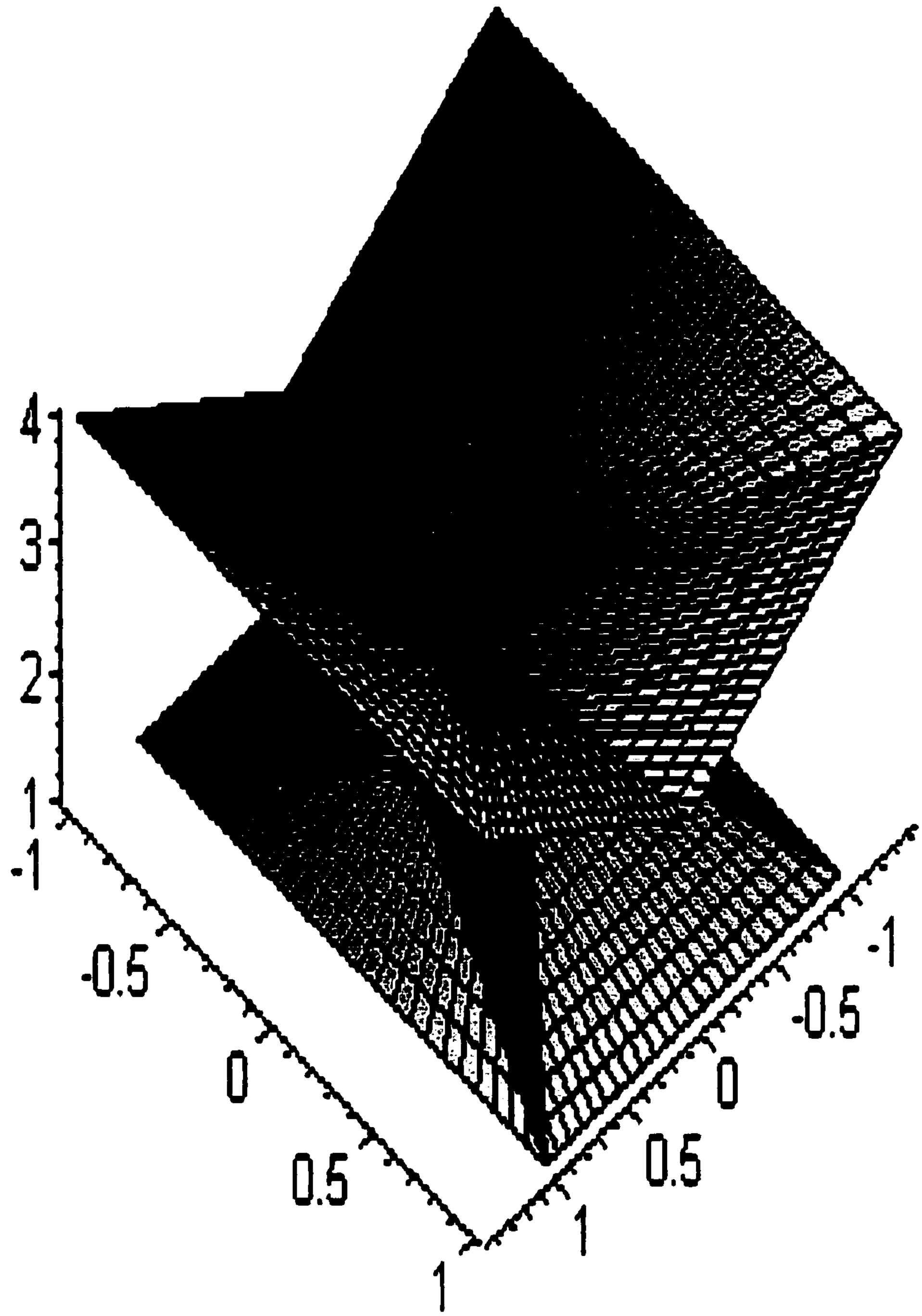

Figure 6.6: Envelope for Distant Cubes Rotated an Angle of $\pi/4$ About y-axis

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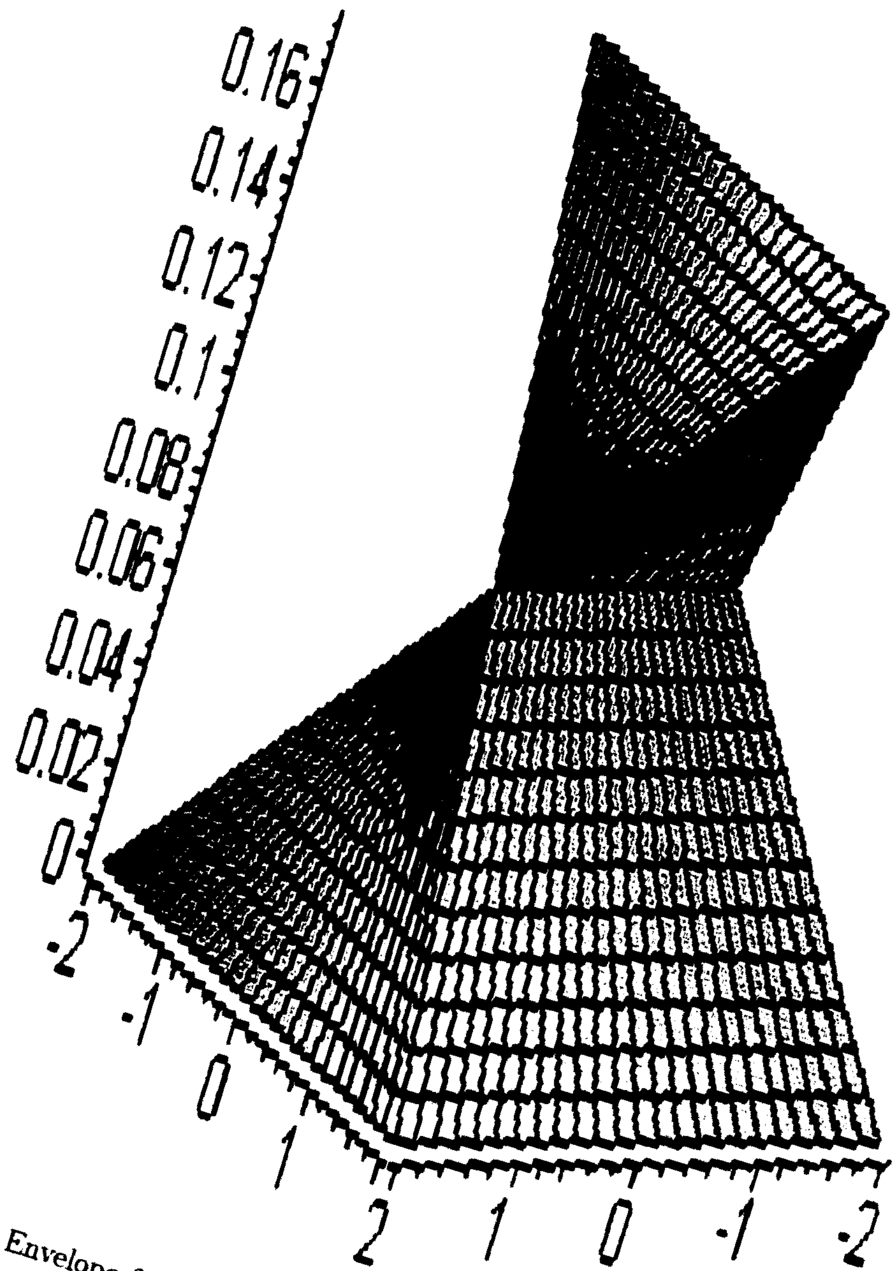

Figure 6.7: Envelope for Close Cubes Rotated an Angle of $\pi/4$ About y-axis

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This program computes the probability that a random plane separates
two fixed disjoint cubes with equal sides s and such that the second
cube has fixed orientation [_,_,_] and fixed center [c1,c2,c3] relative
to the first cube given that the plane hits the fixed sphere centered
at [c1/2,c2/2,c3/2] with radius c+SQR[3]*s where c is the distance
the centers of the two cubes.

Please type a number to represent the common value s for the lengths
of the sides of the two cubes, for example 10, and then press enter.
[s must be greater than 0]
? 10
Please type three numbers separated by commas to represent the
orientation [_,_,_] of the second cube in π radians relative to the
first cube, for example .25,.25,.25, and then press the enter key.
([_,_,_] must satisfy 0_<.5 and 0_<1 and 0_<2]
? .25,.25,.25
Please type three numbers separated by commas to represent the
center [c1,c2,c3] of the second cube relative relative to the
the first cube, for example 0,20,0, and then press the enter key.
The two cubes must not intersect.
? 0,20,0

Here is the cube pair you have chosen. In the drawing the cube pair
is rotated so that the center of each cube is on the vertical axis.

Theoretical computation of probability may take several minutes.
Please type 1 to skip or any other number not to skip.
Do you wish to skip to empirical computation of probability?

Figure 6.8: Q-Basic Input for Cubes

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The motion-invariant measure of planes separating two cubes is the sum of measures of the solitary separating wedges with a minus or plus sign depending on whether the needle of the wedge is an edge of a cube or not respectively plus the sum of the clustered separating wedges such that the vertices of the needle of the wedge are from different cubes. The measure of the wedge is half of the length of the needle times the size of the angle. See Ambartzumian's red book page 114 for analogous formula for convex polyhedrons.

The above measure is converted to a probability measure by dividing by the measure of planes hitting a sphere containing the two cubes. This measure is 2πd where d is the diameter of the sphere containing the two cubes. See Ambartzumian's red book page 122.

We somewhat arbitrarily take d to be c+SQR(3)*s.

Computing theoretical probability . . . very roughly 6 % completed

The standard separating measure = 4.435766

The cubes are contained in a sphere of radius 18.66025
Thus we take our theoretical probability to be the conditional probability that a plane separates the cubes given that it hits this sphere. Thus the theoretical

probability = measure / (4 * pi * r) = 1.891652E-02

Please type a number N to represent the number of pairs of cubes to be randomly placed on a grid of parallel planes in order to compute the empirical probability that a plane separates the cubes. (N must be a positive integer.) How many pairs of cubes? 100

Figure 6.9: Q-Basic Theoretical Results

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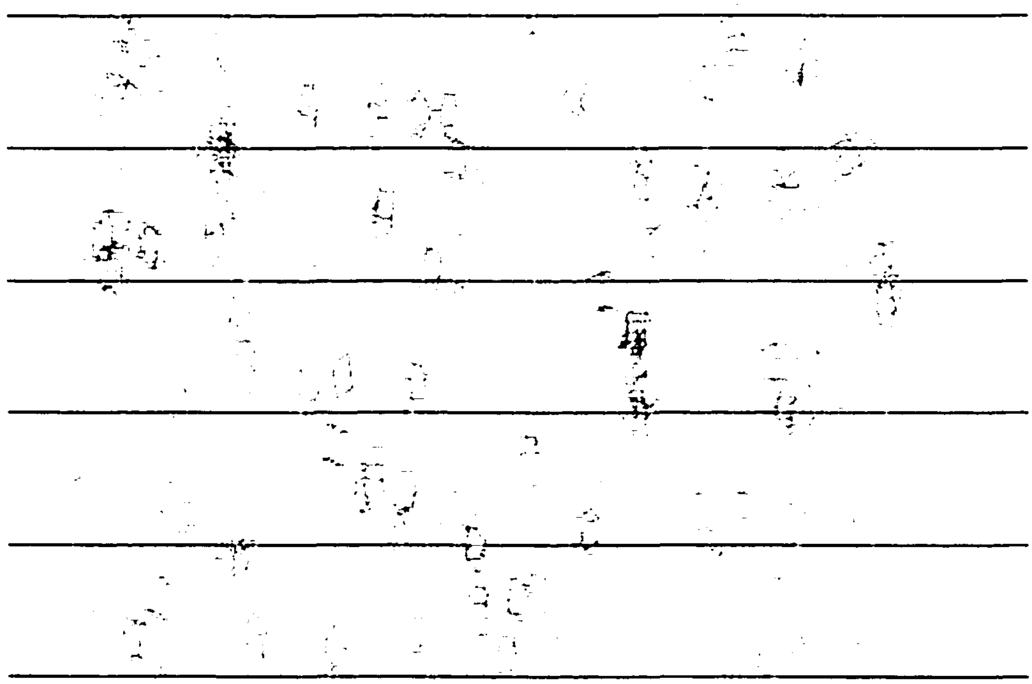
```
 1 separation(s) in  100 tries.
Empirical probability = .01
To see the cubes clearly turn up the brightness and contrast.
```



```
Press any key to continue.  Are you ready to continue?
```

```
 1 separation(s) in  100 tries.
Empirical probability = .01
Estimated standard error = 9.949874E-03
Theoretical probability = 1.891652E-02

Press any key to continue
```

Figure 6.10: Q-Basic Empirical Results

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CHAPTER 7

CONICAL ENVELOPES

7.1 Introduction

In general the separating double support planes for a pair of convex bodies share no point in common and thus the envelope is not conical. There are however interesting examples where the envelope is conical. If the envelope is conical then the computations are easier. A helpful first step in analyzing such situations is to be able to compute the measure of planes intersecting the union of a convex body and the adjoining part of the envelope up to the vertex. Thus we want to compute the measure of the convex hull of the union of a point and a convex body. In other words we want to be able to compute the measure of a coned convex body.

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7.2 Measure of Planes Intersecting a Coned Convex Body

A coned convex body is the convex hull of the union of a convex body and a point. If the original convex body is polyhedral then the coned convex body will also be polyhedral. In such a case the measure of planes intersecting the coned body is known to be equal to the sum of the wedge functions over the edges of the coned body where the wedge function is equal to half the length of the edge times the outer angle of the wedge. See for example Ambartzumian (1990, 114). If the original body is smooth then the coned body will be smooth everywhere except perhaps the vertex and the points of tangency. See Figure 7.1 below for examples of coned convex bodies.

Definition 7.2.1 *A* **smooth cone** *is the convex hull of the union of a smooth convex body and a point.*

We note that the class of smooth cones is more general than the class of smooth convex bodies because the point could be taken to be an interior point.

Ambartzumian (1990, 120-122) gives a proof of a theorem that measure of planes intersecting a smooth compact strictly convex body in R^3 is equal to the integral of absolute mean curvature over the boundary of the body. Ambartzumian's assumption that the body is strictly convex was not explicit but

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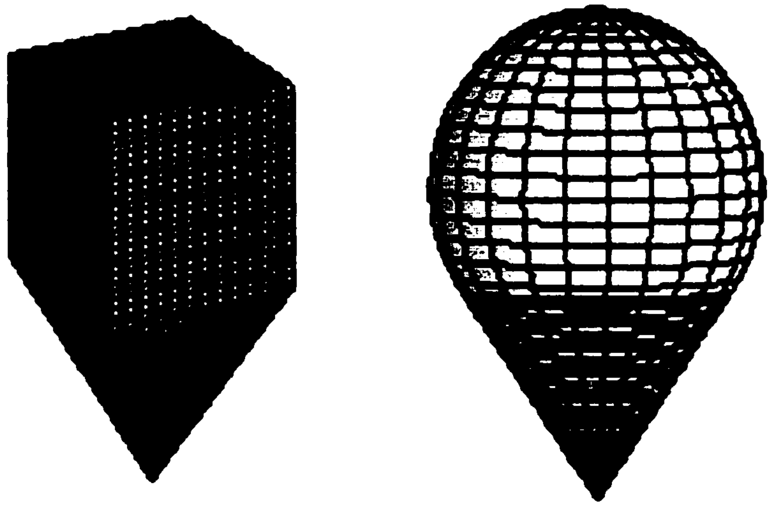

Figure 7.1: Coned Cube and Coned Sphere

implied by the fact that the result was expressed in terms of principle radii of curvature. Ambartzumian states that the theorem is well-known. R. Deltheil (1926, 95) attributes the theorem to H. Minkowski. Below we essentially repeat the proof found in Ambartzumian's book with some modifications so that the result applies to smooth cones.

Lemma 7.2.2 *Almost every plane which intersects the interior of a compact strictly convex smooth cone intersects the boundary of the smooth cone in a continuous simple closed curve which is smooth except at the points of the boundary between the smooth part and the cone and which is continuously differentiable everywhere.*

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Proof. The proof will be in two steps.

Step 1. We will show that the tangent points form a smooth curve. Let A denote a compact smooth strictly convex body in R^3. Let (x, y, z) denote a point in R^3. Let B denote the convex hull of the union of A with a point not in A. Without loss of generality we may locate the vertex of B at the origin, locate the closest point of A on the z-axis, and parametrize the boundary of A by direction (θ, ϕ) of the outward normal vector for $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$ where ϕ is the angle of the normal vector with the z-axis and θ is the angle with the x-axis of the projection of the normal vector onto the xy-plane. Then planes tangent to the conical portion of the boundary of B must satisfy

$$N(\theta, \phi) \cdot S(\theta, \phi) = 0$$

where $S(\theta, \phi)$ is a point on the boundary of A and $N(\theta, \phi)$ is the unit normal vector at that point. Let $f : R^2 \to R$ be defined by $f(\theta, \phi) = N(\theta, \phi) \cdot S(\theta, \phi)$. Then the partial derivative

$$f_\phi = N_\phi(\theta, \phi) \cdot S(\theta, \phi) + N(\theta, \phi) \cdot S_\phi(\theta, \phi) = N_\phi(\theta, \phi) \cdot S(\theta, \phi).$$

In order to apply the implicit function theorem (Theorem 1.7.1) we wish to show that f_ϕ is non-vanishing. We proceed as follows.

Because S is parametrized by normal direction (θ, ϕ) the equation of the normal vector at a point $S(\theta, \phi)$ is

$$N(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

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Thus differentiating N yields the partial derivatives

$$N_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) \text{ and}$$

$$N_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi).$$

Thus N, N_θ, and N_ϕ are mutually orthogonal. Thus N_θ and N_ϕ may be identified with a pair of orthogonal vectors in the tangent plane at $S(\theta, \phi)$. We also note that the z-component of N_θ is zero. The line segment through the origin and from a point of tangency $S(\theta, \phi)$ may also be identified with a tangent vector v. We note that the z component of v is nonzero by construction. Thus v is coplanar with N_θ and N_ϕ and not parallel to N_θ. Therefore v is not perpendicular to N_ϕ. Also inspection of the formula shows N_ϕ is nonzero. Thus f_ϕ is non-vanishing. Therefore by the implicit function theorem there exists a smooth function $\psi(\theta)$ satisfying

$$N(\theta, \psi(\theta)) \cdot S(\theta, \psi(\theta)) = 0.$$

Thus the cone and the original surface share points of tangency on the smooth curve $S(\theta, \psi(\theta))$. Furthermore the conical part of the smooth cone may be parametrized by $tS(\theta, \psi(\theta))$ for $0 \leq t \leq 1$ and thus is smooth away from the vertex and the double tangent curve. Also at the double tangent curve the smooth cone is continuously differentiable since the tangent planes for both surfaces agree at those points. Thus the boundary of the smooth cone is continuously differentiable except at the vertex.

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Step 2. We will now characterize the intersection of a plane with the smooth cone. Let T denote the boundary of the smooth cone. Parametrize T by

$$T(\theta, \phi) = \begin{cases} \frac{\phi}{\psi(\theta)} S(\theta, \psi(\theta)) & \text{for } 0 \leq \phi \leq \psi(\theta) \\ S(\theta, \phi) & \text{for } \psi(\theta) \leq \phi \leq \pi \end{cases}.$$

Let P denote a plane intersecting the interior of the smooth cone. Let u denote a unit normal vector to P. Assume the z-coordinate of u is nonzero. Let d denote the signed distance of the plane from the origin. Assume $d \neq 0$. We note that the set of planes which intersect the interior of the smooth cone but which do not satisfy these conditions on d and the z-coordinate of u is a set of measure zero.

Then the intersection of P and T must satisfy $u \cdot T(\theta, \phi) = d$. Let $f(\theta, \phi) = u \cdot T(\theta, \phi)$. Then f is continuously differentiable except at the vertex and

$$f_\phi = u \cdot T_\phi = \begin{cases} u \cdot \frac{1}{\psi(\theta)} S & \text{for } 0 \leq \phi \leq \psi(\theta) \\ u \cdot S_\phi & \text{for } \psi(\theta) \leq \phi \leq \pi \end{cases}.$$

Thus for $0 < \phi \leq \psi(\theta)$ since P is assumed to be a plane which does not go through the origin then f_ϕ is nonzero. Also for $\psi(\phi) \leq \phi \leq \pi$ since u has a nonzero vertical component then it is not parallel to S_θ and thus not perpendicular to S_ϕ. Thus in either case f_ϕ is nonvanishing. Thus by the implicit function theorem there is a continuously differentiable function $\zeta(\theta)$

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satisfying $u \cdot T(\theta, \zeta(\theta)) = d$. Thus the intersection of P and T is given by the continuously differentiable curve $T(\theta, \zeta(\theta))$.

Furthermore since f is smooth except where $\phi = 0$ or $\varphi = \psi(\theta)$ the implicit function theorem implies that the curve $T(\theta, \zeta(\theta))$ is smooth on the smooth part of the boundary.

Furthermore since a plane and the smooth cone are both convex their intersection will be convex. Since the plane is assumed to intersect the interior of the smooth cone, the intersection will be a planar convex body. Thus the boundary of the intersection will be a simple closed curve. Thus the intersection of P and T will be a simple closed curve which is continuously differentiable everywhere smooth everywhere except at points of the tangent curve and differentiable everywhere. □

We can now use this lemma to express the measure of planes intersecting a smooth cone in terms of mean curvature. The proof is similar to a proof found in Ambartzumian's book (1990, 120-122) but with some modification to allow for the presence of a singularity.

Theorem 7.2.3 *The measure of planes intersecting a smooth cone in R^3 is equal to the integral of absolute mean curvature over the boundary of the body.*

Proof. Let A denote a smooth cone in R^3. Let (x, y, z) denote a point in R^3. Coordinatize planes by angle φ of the normal vector with the z-axis, angle θ

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with the x-axis of the projection onto the xy-plane of the normal vector, and signed distance ρ from the origin. In these coordinates the measure of planes intersecting A is given by Theorem 1.5.5 as

(7.2.4)
$$\int_0^{2\pi}\int_0^{\pi/2}\left[\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} d\rho\right]\sin\phi\, d\phi\, d\theta.$$

Let P_C be a plane intersecting the boundary of A in a curve C. Then by Lemma 7.2.2 above except for a set of planes of measure zero the points C of intersection form a simple closed curve which is smooth everywhere except points on the boundary between the strictly convex part and the cone and differentiable everywhere.

Let ds denote the length element along C. Let $k(\rho,\phi,\theta,s)$ denote the curvature on smooth portions of the curve. Let $I = I(\rho,\phi,\theta)$ denote the indicator function on the set of planes which intersect S in a simple closed curve which is smooth everywhere except points on the boundary between the strictly convex part and the cone and differentiable everywhere. Let $J(\rho,\phi,\theta,s)$ denote the indicator function on the set of non-smooth points of C. Let $\alpha = \alpha(\rho,\phi,\theta,s)$ denote the exterior angle at a point of C. We note that for almost all planes $\alpha = 0$ at all points of C including points on the boundary between the strictly convex part and the cone. Then according to the Theorem of Turning Tangents

$$\frac{1}{2\pi}\left(IJ(\rho,\phi,\theta,s)\alpha(\rho,\phi,\theta,s) + \left[\int_{s_1(\rho,\phi,\theta)}^{s_2(\rho,\phi,\theta)} I(\rho,\phi,\theta)k(s,\rho,\phi,\theta)ds\right]\right) = 1.$$

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where $s_2 - s_1$ is the length of the curve. See for example Do Carmo (1976, 267). Then Lemma 7.2.2 implies that except on a set of planes of measure zero $I = 1$ and $\alpha = 0$. Thus except on a set of measure zero

(7.2.5)
$$\frac{1}{2\pi}\int_{s_1(\rho,\phi,\theta)}^{s_2(\rho,\phi,\theta)} k(s,\rho,\phi,\theta)ds = 1.$$

Then inserting Equation 7.2.5 into Equation 7.2.4 yields

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\left[\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)}\int_{s_1(\rho,\phi,\theta)}^{s_2(\rho,\phi,\theta)} k(s,\rho,\phi,\theta)dsd\rho\right]\sin\phi\ d\phi\ d\theta.$$

At each point of the curve C consider the plane P_D perpendicular to both the tangent plane and the plane of C. Then Lemma 7.2.2 implies this plane almost everywhere intersects the boundary of A in a simple closed piecewise smooth curve D. Let dt represent the length element along curve D. Let γ denote the angle between the direction of dt and the direction of $d\rho$. Then a change of variable $d\rho = \cos\gamma\ dt$ yields

(7.2.6)
$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\left[\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)}\int_{s_1(\rho,\phi,\theta)}^{s_2(\rho,\phi,\theta)} k(s,\rho(t),\phi,\theta)\cos\gamma\ ds\ dt\right]\sin\phi\ d\phi\ d\theta.$$

According to a theorem of Meusnier we can replace the factor $k(s,\rho(t),\phi,\theta)\cos\gamma$ appearing in the integrand above with the normal curvature $k_n(s,t,\phi,\theta)$ to get

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\left[\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)}\int_{s_1(\rho,\phi,\theta)}^{s_2(\rho,\phi,\theta)} k_n(s,t,\phi,\theta)\ ds\ dt\right]\sin\phi\ d\phi\ d\theta.$$

See Ambartzumian (1990, 122). Let $k_1(s,t)$ and $k_2(s,t)$ denote the principal curvatures at the point (s,t) and let $\zeta_1(s,t,\phi,\theta)$ and $\zeta_2(s,t,\phi,\theta)$ represent the

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angles of the normal direction above with the respective principle directions. Then we can use the Euler formula to replace the normal curvature in the formula above with an expression involving the principal curvatures. This yields

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\left[\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)}\int_{s_1(\rho,\phi,\theta)}^{s_2(\rho,\phi,\theta)}\right.$$

$$\left.\Big(k_1(s,t)\cos^2(\zeta_1(s,t,\phi,\theta)) + k_2(s,t)\cos^2(\zeta_2(s,t,\phi,\theta))\Big)\ ds\ dt\right]\sin\phi\ d\phi\ d\theta.$$

See Do Carmo (1976, 145). We next note that the integration limits of the inner two integrals are over the entire body and thus do not depend on ϕ and θ. Thus we can replace $ds\ dt$ with surface area element dS and reverse the order of integration to get

$$\frac{1}{2\pi}\int_A\int_0^{2\pi}\int_0^{\pi/2}\Big(k_1\cos^2(\zeta_1(s,t,\phi,\theta)) + k_2\cos^2(\zeta_2(s,t,\phi,\theta))\Big)\sin\phi\ d\phi\ d\theta\ dS.$$

Next we use a trig identity to rewrite the integral as

$$\frac{1}{2\pi}\int_A\int_0^{2\pi}\int_0^{\pi/2}$$

$$\frac{1}{2}\Big(k_1\big(1+\cos(2\zeta_1(s,t,\phi,\theta))\big) + k_2\big(1+\cos(2\zeta_2(s,t,\phi,\theta))\big)\Big)\sin\phi\ d\phi\ d\theta\ dS.$$

When integrating the above equation over all values of ϕ and θ the values of ζ_1 and ζ_2 will be evenly distributed over all values between 0 and 2π. Thus the cos terms will be zero. A straightforward integration of the remaining terms with respect to ϕ and θ then yields

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$$\int_A \left(\frac{k_1 + k_2}{2} \right) dS$$

which is what was to be proved. □

7.3 Measure of Planes Separating a Convex Body and a Point

Lemma 7.3.1 *Let A represent a point in R^3. Let B be a disjoint compact convex body in R^3. Let C denote the convex hull of $A \cup B$. Then the set of planes which separate A and B is the set of planes which intersect C but not $A \cup B$.*

Proof. Let P be a plane which separates A and B. Let H_A denote the closed half-space determined by P which contains A. Let H_B denote the open half-space determined by P which contains B. Since P separates A and B there are distinct points $a \in A \subset H_A$ and $b \in B \subset H_B$. Since C is convex it contains the line segment $(1-t)a+tb$ for $0 \leq t \leq 1$ connecting a and b. Let h_A and h_B represent the intersections of H_A and H_B respectively with the line segment. Since H_A and H_B are connected, disjoint, and mutually exclusive, so are h_A and h_B. Thus the line segment is a union of two disjoint mutually exclusive connected sets. Furthermore h_A is closed and bounded. Thus there is a largest value t_0 of t such that $(1-t)a+tb \in h_A$. Thus P intersects C at $(1-t_0)a+t_0b$.

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Thus the set of planes which separate A and B is a subset of the set of planes which intersect C but not $A \cup B$.

Now let P represent a plane which intersects C but not $A \cup B$. Then since C is convex there exists points $a \in A$ and $b \in B$ and t such that $0 \leq t \leq 1$ and such that $(1-t)a + tb \in P \cap C$. Thus P separates a and b. Since P does not intersect $A \cup B$ then all points of B must be in the same half-plane as b. Thus P separates B and A. Thus the set of planes which intersect C but not $A \cup B$ is a subset of the set of planes which separate A and B. Thus the set of planes which separate A and B is the set of planes which intersect C but not $A \cup B$. □

Theorem 7.3.2 *Let A represent a point in R^3. Let B be a disjoint compact strictly convex body in R^3. Let C denote the convex hull of $A \cup B$. Then the measure of planes separating A and B is equal to the integral of mean curvature over the boundary of C minus the integral of mean curvature over the boundary of B.*

Proof. By the lemma above the measure of planes which separate A and B is equal to the measure of planes which intersect C but not $A \cup B$. Since the measure of planes through a point is zero. this is equal to the measure of planes which intersect C but not B. Since $B \subset C$ this is equal to the measure of planes which intersect C minus the measure of planes which intersect B. By

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Theorem 7.2.3 above this is equal to the total absolute mean curvature over the boundary of C minus the total absolute mean curvature over the boundary of B. □

7.4 Other Conical Envelopes

Theorem 7.4.1 *Let A_1 and A_2 be disjoint compact convex bodies in R^3. Assume that the separating double support planes meet in exactly one point p. For $i = 1, 2$ let B_i denote the convex hull of $A_i \bigcup \{p\}$. Also let $[A_i]$ and $[B_i]$ denote the sets of planes intersecting the interiors of A_i and B_i respectively. Let $[A_1||A_2]$ denote the set of planes which separate A_1 and A_2. Let $[co\ A_1 \bigcup A_2]$ denote the set of planes which intersect the convex hull of $A_1 \bigcup A_2$. Let m denote a measure on the set of planes in R^3 which is invariant under rigid motions. Then except perhaps for a set of measure zero*

(i) $[A_1] \bigcap [A_2] = [B_1] \bigcap [B_2]$.

(ii) $[B_1] \bigcup [B_2] = [co\ A_1 \bigcup A_2]$.

(iii) $[A_1||A_2] = [co\ A_1 \bigcup A_2] \backslash [A_1 \bigcup A_2]$.

Proof of (i). Note that for $i = 1, 2$ $[A_i] \subset [B_i]$ because $[B_i]$ is the convex hull of $[A_i]$ and a point. Thus $[A_1] \bigcap [A_2] \subset [B_1] \bigcap [B_2]$. Thus we only need to show that $[B_1] \bigcap [B_2] \subset [A_1] \bigcap [A_2]$.

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Let P be a plane which intersects B_1 and B_2 at points b_1 and b_2 respectively. Since we are not concerned with sets of measure zero we can assume that P does not intersect p. Then for $i = 1, 2$ b_i is on the line segment a_ip for some point $a_i \in A_i$. Without loss of generality assume b_i is an interior point of the line segment. Otherwise we are done. Also without loss of generality we can take b_i to be on the boundary of B_i. Thus the line segment a_2p can be extended to a separating double support line containing a point $a_1' \in A_1$. a_1' is distinct from a_1. Otherwise b_1 and b_2 would be on that same line through p and thus P would intersect p. Thus the line through b_1b_2 intersects the triangle $a_1a_1'p$ on the interior of side a_1p at the point b_1. This line must intersect one of the other sides of the triangle. It cannot intersect the side $a_1'p$. Otherwise b_1 and b_2 would lie on the same line through p and thus P would intersect p. Therefore the line through b_1b_2 must intersect the side a_1a_1'. The point of intersection will lie in A_1 since A_1 is convex. Thus P contains a point of A_1. Thus $P \in [A_1]\bigcap[A_2]$. But P was an arbitrary element of $[B_1]\bigcap[B_2]$ excluding a set of measure zero. Thus $[B_1]\bigcap[B_2] \subset [A_1]\bigcap[A_2]$ except for a set of measure zero. Thus $[B_1]\bigcap[B_2] = [A_1]\bigcap[A_2]$ except for a set of measure zero. This proves (i).

Proof of (ii). We note that p is in the convex hull of $A_1 \bigcup A_2$ since it connects points of double tangency. Thus for $i = 1, 2$, B_i which is the convex

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hull of $A_i \bigcup p$ is a subset of the convex hull of $A_1 \bigcup A_2$. Thus $[B_1] \bigcup [B_2] \subset [co\ A_1 \bigcup A_2]$. Thus we only need to show that $[co\ A_1 \bigcup A_2] \subset [B_1] \bigcup [B_2]$.

Let P be a plane which intersects the convex hull of $A_1 \bigcup A_2$ at a point c. Without loss of generality assume that $c \notin A_1 \bigcup A_2$. Otherwise we are done. Then for $i = 1, 2$ there are points $a_i \in A_i$ such that c lies on the interior of the line segment $a_1 a_2$. Without loss of generality assume that a_1, a_2, and p are not collinear. Otherwise we are done. Since P intersects the triangle $a_1 a_2 p$ at an interior point c on side $a_1 a_2$ then P must intersect one of the other sides of the triangle. But each of the other sides is contained in one of the B_i. Thus P intersects one of the B_i. Thus an arbitrary plane $P \in [co\ A_1 \bigcup A_2]$ is contained in $[B_1] \bigcup [B_2]$. Thus $[co\ A_1 \bigcup A_2] \subset [B_1] \bigcup [B_2]$. Thus $[co\ A_1 \bigcup A_2] = [B_1] \bigcup [B_2]$. This proves (ii).

Proof of (iii). We note that if a plane P separates A_1 and A_2 then by definition P doesn't intersect A_1 or A_2 and furthermore A_1 and A_2 are in different half-spaces determined by P. Thus for $i = 1, 2$ there are points $a_i \in A_i$ such that P intersects the line segment $a_1 a_2$. Thus P intersects the convex hull of $A_1 \bigcup A_2$ but not A_1 or A_2. Thus $[A_1 || A_2] \subset [co\ A_1 \bigcup A_2] \backslash [A_1 \bigcup A_2]$ and we need only show that $[co\ A_1 \bigcup A_2] \backslash [A_1 \bigcup A_2] \subset [A_1 || A_2]$.

Let $P \in [co\ A_1 \bigcup A_2] \backslash [A_1 \bigcup A_2]$. Suppose two points $a_{10}, a_{11} \in A_1$ lie in different half-spaces determined by P. Then since A_1 is convex the entire line

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segment $a_{10}a_{11}$ is contained in A_1. Furthermore P intersects this line segment. Thus P intersects A_1. This is a contradiction. Therefore all of A_1 lies in the same half-space determined by P. Likewise all of A_2 lies in the same half-space determined by P.

Now suppose that A_1 and A_2 both lie in the same half-space determined by P. Then the convex hull of $A_1 \bigcup A_2$ lies in that half-space. Then P does not intersect the convex hull of $A_1 \bigcup A_2$. This is a contradiction. Thus A_1 and A_2 lie in different half-spaces determined by P. Thus P separates A_1 and A_2. Thus $[co\ A_1 \bigcup A_2] \backslash [A_1 \bigcup A_2] \subset [A_1 || A_2]$. Thus $[co\ A_1 \bigcup A_2] \backslash [A_1 \bigcup A_2] = [A_1 || A_2]$. This proves (iii). □

Remark 7.4.2 *The arguments are almost the same when considering lines separating a pair of convex bodies in R^2. However previous authors Crofton* (1869) *and Sylvester* ([1890] 1973) *seemed to implicitly assume these results in R^2. A search of the literature did not turn up an explicit proof of these results.*

Theorem 7.4.3 *Let A_1 and A_2 be disjoint compact convex bodies in R^3. Assume that the separating double support planes meet in exactly one point p. For $i = 1, 2$ let B_i denote the convex hull of $A_i \bigcup \{p\}$. Also let $[A_i]$ and $[B_i]$ denote the sets of planes intersecting the interiors of A_i and B_i respectively. Let $[A_1 || A_2]$ denote the set of planes which separate A_1 and A_2. Let $[co\ A_1 \bigcup A_2]$ denote the set of planes which intersect the convex hull of $A_1 \bigcup A_2$. Let m*

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denote a measure on the set of planes in R^3 which is invariant under rigid motions. Then except perhaps for a set of measure zero

(i) $m[A_1]\bigcap[A_2] = m[B_1] + m[B_2] - m[co\ A_1 \bigcup A_2]$.

(ii) $m[A_1]\bigcup[A_2] = m[co\ A_1 \bigcup A_2] + m[A_1] + m[A_2] - m[B_1] - m[B_2]$.

(iii) $m[A_1||A_2] = m[B_1] + m[B_2] - m[A_1] - m[A_2]$.

Proof of (i). $m[A_1]\bigcap[A_2] = m[B_1]\bigcap[B_2]$ by Theorem 7.4.1(i)

$= m[B_1] + m[B_2] - m[B_1]\bigcup[B_2]$ from measure theory

$= m[B_1] + m[B_2] - m[co\ A_1 \bigcup A_2]$ substituting the result from Theorem 7.4.1(ii) for $[B_1]\bigcup[B_2]$ above.

Proof of (ii). $m[A_1]\bigcup[A_2] = m[A_1] + m[A_2] - m[A_1]\bigcap[A_2]$ from measure theory.

$= m[co\ A_1 \bigcup A_2] + m[A_1] + m[A_2] - m[B_1] - m[B_2]$ substituting results from part (i) above for $m[A_1]\bigcap[A_2]$.

Proof of (iii). $m[A_1||A_2] = m[co\ A_1 \bigcup A_2]\backslash[A_1 \bigcup A_2]$ from Theorem 7.4.1 above

$= m[co\ A_1 \bigcup A_2] - m[A_1 \bigcup A_2]$ by additivity of measures

$= m[B_1] + m[B_2] - m[A_1] - m[A_2]$ substituting results from part (ii) above for $m[A_1 \bigcup A_2]$. □

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Throughout this paper the main purpose of the assumption of compactness is to assure the existence of the envelope. We note that if A_1 and A_2 are not compact but nevertheless have a conical envelope then we may replace A_1 and A_2 with compact A_1' and A_2' which have the same caps and envelope and thus the same separating planes and apply formula (iii) above. Thus the assumption of compactness may be removed for the following corollaries.

Corollary 7.4.4 *Let A and B be disjoint polyhedra with positive volume in R^3 and with a conical envelope of separating double support planes. Then the measure of planes separating A and B is equal to the wedge function over the envelope minus the wedge function over the caps.*

Corollary 7.4.5 *Let A and B be disjoint strictly convex bodies in R^3 and with a conical envelope of separating double tangent planes. Then the measure of planes separating A and B is equal to the integral of absolute mean curvature over the envelope minus the integral of absolute mean curvature over the caps.*

Theorem 7.4.6 *If A is a strictly convex body in R^3 and if B is disjoint from A and the symmetric image of A with respect to a point then the envelope of separating double tangent planes if it exists is a cone with a vertex at that point and the measure of planes separating A and B is the total absolute mean curvature over the envelope minus the total absolute mean curvature over the caps.*

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Proof. Without loss of generality parametrize the nearby boundaries s_A and s_B of A and B respectively as follows.

$s_A(x,y) = (x,y,-f(x,y))$, $s_B(x,y) = (-x-y,c+f(x,y))$ where $c > 0$ and f is a strictly convex function defined on a bounded region which includes the interior point $0,0$. Furthermore assume that f has vertical tangent planes on the boundary of the region and that $f(0,0) = f'(0,0) = 0$. Thus the pair of surfaces is symmetric with respect to the point $(0,0,c/2)$.

A straightforward computation of the normal vectors to the surfaces yields

$$n_A(x_A,y_A) = \frac{(f_x(x_A,y_A),f_y(x_A,y_A),1)}{\sqrt{f_x(x_A,y_A)^2 + f_y(x_A,y_A)^2 + 1}} \text{ and}$$

$$n_B(x_B,y_B) = \frac{(f_x(x_B,y_B),f_y(x_B,y_B),1)}{\sqrt{f_x(x_B,y_B)^2 + f_y(x_B,y_B)^2 + 1}}.$$

Solving $n_A = n_B$ yields $(x_A,y_A) = (x_B,y_B)$. Thus the tangent planes of the two surfaces are parallel when the (x,y) coordinates are the same.

Solving $n_A \cdot s_A = n_A \cdot (0,0,z_A)$ for z_A yields

$$z_A = xf_x + yf_y - f.$$

Similarly solving $n_B \cdot s_B = n_B \cdot (0,0,z_B)$ for z_B yields

$$z_B = -xf_x - yf_y + f + c.$$

Solving $z_A = z_B$ yields $z_A = z_B = c/2$.

Thus the separating double tangent planes meet at the point $(0,0,c/2)$. Therefore by Corollary 7.4.5 above the measure of planes separating the two

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surfaces is the total absolute mean curvature of the envelope minus the total absolute mean curvature of the caps. □

In Chapters 9 and 10 we will examine some particular examples of pairs of surfaces with conical envelopes in greater detail.

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CHAPTER 8

SMOOTH CONVEX BODIES: PRELIMINARY RESULTS

8.1 Introduction

A classical result in integral geometry is that the measure of planes intersecting a compact strictly convex body in R^3 is equal to the integral of absolute mean curvature over the surface of the body. See for example Ambartzumian (1990, 120-122). This theorem inspired us to try to show that the measure of planes separating two disjoint compact smooth convex bodies in R^3 is equal to the integral of absolute mean curvature over the envelope of separating double tangent planes minus the integral of absolute mean curvature over the caps. In other chapters we prove this conjecture for certain special cases. The re-

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mainder of this chapter represents our progress on and ideas for solving the general case.

8.2 The Direct Approach

One possible approach to the general case that has worked well in some of the special cases is to compute the integral of mean curvature and to compute the separating measure and then try to equate the two integrals. Theorems 1.5.4 and 1.5.5 of Chapter 1 may be applied to directly integrate the measure of planes separating two strictly convex bodies in R^3 as follows.

Theorem 8.2.1 *Parametrize two disjoint compact strictly convex bodies A and B such that the z-axis contains a shortest line segment connecting A and B and such that the closest points are $(0,0,0) \in A$ and $(0,0,c) \in B$ where $c > 0$. Let $\mathbf{s_A}$ and $\mathbf{s_B}$ be the boundaries of A and B respectively. Let $\mathbf{n_A}(\theta, z)$ denote the outward and $\mathbf{n_B}(\theta, z)$ denote the inward unit normal vectors of $\mathbf{s_A}$ and $\mathbf{s_B}$ respectively. Parametrize $\mathbf{s_A}$ and $\mathbf{s_B}$ by angle θ of the x-axis with the projection of the normal vector onto the xy-plane and intersection z of the tangent plane with the z-axis. Let $z(\theta)$ be the unique solution of*

$$\mathbf{n_A}(\theta, z) \cdot \mathbf{s_A}(\theta, z) = \mathbf{n_B}(\theta, z) \cdot \mathbf{s_B}(\theta, z).$$

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Then the measure of planes separating A and B is

$$\frac{1}{2}\int_0^{2\pi}\int_0^{z(\theta)}\left[\left(\mathbf{n_A}(\theta,z)\cdot(1,0,0)\right)^2+\left(\mathbf{n_A}(\theta,z)\cdot(0,1,0)\right)^2\right]\,dz\,d\theta$$
$$+\frac{1}{2}\int_0^{2\pi}\int_{z(\theta)}^{c}\left[\left(\mathbf{n_B}(\theta,z)\cdot(1,0,0)\right)^2+\left(\mathbf{n_B}(\theta,z)\cdot(0,1,0)\right)^2\right]\,dz\,d\theta.$$

Proof. A necessary condition for two planes to be the same is that their normal vectors be parallel. Furthermore by construction in order for a double tangent plane to be separating the normals must be the same. Because the surfaces are assumed to be parametrized by tangent planes this means that

$$\mathbf{n_A}(\theta_A,z_A)=\mathbf{n_B}(\theta_B,z_B).$$

which implies that $\theta_A=\theta_B$ and $z_A=z_B$. Thus we can drop the subscripts. A second necessary condition that two tangent planes are the same is that

$$\mathbf{n_A}(\theta,z)\cdot\mathbf{s_A}(\theta,z)=\mathbf{n_B}(\theta,z)\cdot\mathbf{s_B}(\theta,z).$$

Theorem 4.7.1 implies that there is exactly one such tangent plane for each angle θ. Thus the above equation has a unique solution $z(\theta)$.

We parametrize planes in R^3 by angle ϕ of the upward normal with the z-axis, by abuse of notation z-intercept z, and by angle θ with the x-axis of the projection of the upward normal onto the xy-plane. According to Theorem 1.5.4 the invariant measure in these coordinates is $\cos\phi\sin\phi\;d\phi\;dz\;d\theta$.

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For fixed θ and for fixed $z < z(\theta)$ the separating planes will be bounded by tangent planes to A. The ϕ-coordinate of such tangent planes will be

$$\cos^{-1}\Big(\mathbf{n_A}(\theta, z) \cdot (0, 0, 1)\Big).$$

Likewise for fixed θ and for fixed $z > z(\theta)$ the separating planes will be bounded by tangent planes to B. The ϕ-coordinate of such tangent planes will be

$$\cos^{-1}\Big(\mathbf{n_B}(\theta, z) \cdot (0, 0, 1)\Big).$$

Thus the measure of planes separating A and B is

$$\int_0^{2\pi}\int_0^{z(\theta)}\int_0^{\cos^{-1}\left(\mathbf{n_A}(\theta,z)\cdot(0,0,1)\right)} \cos\phi \sin\phi \; d\phi \; dz \; d\theta$$

$$+ \int_0^{2\pi}\int_{z(\theta)}^{r}\int_0^{\cos^{-1}\left(\mathbf{n_B}(\theta,z)\cdot(0,0,1)\right)} \cos\phi \sin\phi \; d\phi \; dz \; d\theta.$$

A straightforward evaluation of the integrals above yields the desired result. □

Theorem 8.2.2 *Let $\mathbf{s_A}$ and $\mathbf{s_B}$ be the boundaries of disjoint compact strictly convex bodies A and B respectively. Let (x, y, z) be a point in R^3. Let the z-axis be through a shortest line segment connecting A and B. Parametrize $\mathbf{s_A}$ by direction (θ, ϕ) of outward and $\mathbf{s_B}$ by direction (θ, ϕ) of the inward normal vector where ϕ is the angle of the normal with the z-axis and θ is the angle with the x-axis of the projection onto the xy-plane of the normal. Let $\mathbf{n}(\theta, \phi)$ denote the unit normal vector. Let $\phi(\theta)$ be the unique solution of*

$$\mathbf{n}(\theta, \phi) \cdot \mathbf{s_A}(\theta, \phi) = \mathbf{n}(\theta, \phi) \cdot \mathbf{s_B}(\theta, \phi).$$

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Then the measure of planes separating A and B is

$$\int_0^{2\pi}\int_0^{\phi(\theta)} \mathbf{n}(\theta,\phi)\cdot(\mathbf{s_B}(\theta,\phi)-\mathbf{s_A}(\theta,\phi))\sin\phi\; d\phi\; d\theta.$$

Proof. A necessary condition for two planes to be the same is that their normal vectors be parallel. Furthermore by construction in order for a double tangent plane to be separating the normals must be the same. Because the surfaces are assumed to be parametrized by tangent planes this means that

$$\mathbf{n_A}(\theta_A,\phi_A)=\mathbf{n_B}(\theta_B,\phi_B).$$

which implies that $\theta_A=\theta_B$ and $\phi_A=\phi_B$. Thus we can drop the subscripts. A second necessary condition that two tangent planes are the same is that

$$\mathbf{n}(\theta,\phi)\cdot\mathbf{s_A}(\theta,\phi)=\mathbf{n}(\theta,\phi)\cdot\mathbf{s_B}(\theta,\phi).$$

Theorem 4.7.1 implies that there is exactly one such tangent plane for each angle θ. Thus the above equation has a unique solution $\phi(\theta)$. In fact using the implicit function theorem in exactly the same way as it was used in Step 1 of the proof of Lemma 7.2.2 one can show that $\phi(\theta)$ is smooth.

We parametrize planes in R^3 by angle ϕ of the upward normal with the z-axis, signed distance ρ from the origin, and by angle θ with the x-axis of the projection of the upward normal onto the xy-plane. According to Theorem 1.5.4 the invariant measure in these coordinates is $\sin\phi\; d\rho\; d\phi\; d\theta$.

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For fixed θ and for fixed $\phi < \phi(\theta)$ the separating planes will be bounded by tangent planes to A and B. The ρ-coordinate of such tangent planes to A will be $\mathbf{n}(\theta, \phi) \cdot \mathbf{s_A}(\theta, \phi)$ and the ϕ-coordinate of such tangent planes to be will be $\mathbf{n}(\theta, \phi) \cdot \mathbf{s_B}(\theta, \phi)$. Thus the measure of planes separating A and B is

$$\int_0^{2\pi} \int_0^{\phi(\theta)} \int_{\mathbf{n}(\theta,\phi)\cdot\mathbf{s_A}(\theta,\phi)}^{\mathbf{n}(\theta,\phi)\cdot\mathbf{s_B}(\theta,\phi)} \sin\phi \; d\rho \; d\phi \; d\theta.$$

A straightforward evaluation of the integrals above yields the desired result. □

For convenience the above theorems assumed compactness but that assumption was somewhat stronger than was necessary. Compactness assures the existence of points of separating double tangency.

The direct computation of the integrals of mean curvature for comparison in the general smooth case proved to be more difficult and less enlightening. However this approach was successful for more specialized cases. This approach was used with some success in proving the main conjecture for surfaces of revolution, a certain fourth order surface, and for a certain class of paraboloids in the following chapters.

8.3 A Minkowski Difference Approach

Another approach that holds some promise for proving the main conjecture is the Minkowski difference approach. The Minkowski difference of two convex bodies A and B is the body formed by performing all possible vector

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subtractions of a point in A from a point in B. We can show that the measure of planes separating two convex bodies in R^3 is equal to the measure of planes separating the Minkowski Difference of the two bodies from the origin. Thus we can reduce the general case to the case of planes separating a convex body and a point which we already dealt with in Chapter 7. Partial results with this approach are given below. The difficult step in this approach is to equate the integrals of mean curvature for the original pair with the integrals of mean curvature for the Minkowski difference and the origin.

We were inspired to try this approach when we came across a theorem that stated that the set of linear functionals separating two convex bodies is the same as the set of linear functionals separating the Minkowski difference from the origin. See Thompson (1996. 47).

Theorem 8.3.1 *Let A and B be convex bodies in R^2 or R^3. Then the boundary of the Minkowski difference may be obtained by subtracting points on the boundaries of A and B with opposite facing outward normal vectors.*

Proof. We prove the theorem for R^3. The proof in R^2 is similar. Fix a direction (θ, ϕ). Let $v(\theta, \phi)$ denote a unit vector with direction (θ, ϕ). Let s_A be a point on the boundary of A. It is a property of convex bodies that each direction is the direction of two support planes. Then because $v \cdot s_A$ gives the signed distance of s_A from the origin, support planes must correspond to the extreme

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values of the signed distance from the origin. Therefore extreme values of the signed distance from the origin occur on the boundaries of A and B. Thus in the direction (θ, ϕ) the extreme values of the Minkowski difference will also be on the boundaries of A and B. The vector points in the direction of increasing values of the signed distance to the origin. Thus the extreme values will occur when the outward normals are opposite facing. □

Theorem 8.3.2 *Let A and B be disjoint compact strictly convex bodies in R^3. Then the measure of planes separating A and B is equal to the measure of planes separating the Minkowski difference M and zero.*

Proof. The set of separating planes are bounded by the set of support planes. Fix a direction (θ, ϕ). Let ρ represent the signed distance from zero. The value of ρ for the planes separating A and B varies from

$$N(\theta, \phi) \cdot S_A(\theta, \phi) \text{ to } N(\theta, \phi) \cdot S_B(\theta, \phi)$$

whereas the value of ρ for the planes separating M from zero varies from

$$N(\theta, \phi) \cdot (0, 0, 0) \text{ to } N(\theta, \phi) \cdot S_M(\theta, \phi)$$

which simplifies to

$$0 \text{ to } N(\theta, \phi) \cdot \left(S_B(\theta, \phi) - S_A(\theta, \phi)\right).$$

Thus the set of planes which separate M and zero is a translation of the set of planes which separate A and B. Thus, since our measure is invariant

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under rigid motions, the measure of planes separating A and B is equal to the measure of planes separating M and zero. □

Lemma 8.3.3 *The radius of curvature on the boundary of the Minkowski difference of two planar disks in parallel planes is equal to the sum of the radii of curvature of the original two bounding circles.*

Proof. Without loss of generality parametrize the circles

$$s_A(\theta) = (r_A \cos\theta_A, r_A \sin\theta_A, 0)$$

$$s_B(\theta) = (a + r_B \cos\theta_B, b + r_B \sin\theta_B, c).$$

By Theorem 8.3.1 the Minkowski difference may be obtained by subtracting points on the circles with opposite facing outward normals. We note that the normal vectors are pointing in opposite directions when $\theta_B = \theta_A + \pi$. Thus we compute

$$s_B(\theta) - s_A(\theta + \pi) = (a + (r_A + r_B)\cos\theta, b + (r_A + r_B)\sin\theta, c).$$

□

Lemma 8.3.4 *The radius of curvature of the Minkowski difference of two strictly convex planar curves in parallel planes is equal to the sum of the radii of curvature of the original two curves.*

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Proof. Label the two curves $\alpha_{\mathbf{A}}$ and $\alpha_{\mathbf{B}}$. Orient $\alpha_{\mathbf{A}}$ with outward normal vectors and $\alpha_{\mathbf{B}}$ with inward normal vectors. Identify parallel directions in the parallel planes. Write $\alpha_{\mathbf{A}}$ and $\alpha_{\mathbf{B}}$ as functions of the angle ν of the normal vector. Since $\alpha_{\mathbf{A}}$ and $\alpha_{\mathbf{B}}$ are strictly convex at a point ν_0 we can approximate $\alpha_{\mathbf{A}}$ and $\alpha_{\mathbf{B}}$ with circles $\mathbf{c_A}$ and $\mathbf{c_B}$ which agree with $\alpha_{\mathbf{A}}$ and $\alpha_{\mathbf{B}}$ respectively at ν_0 up through the second derivative.

Because the radius of curvature depends only on the first two derivatives, the respective radii of curvature also agree. Let r_A and r_B denote the respective radii of curvature.

Let $\alpha_{\mathbf{M}}$ and $\mathbf{c_M}$ denote the Minkowski differences of the original curves and of the two circles respectively. Then by the linearity of the derivative on functions, the Minkowski differences $\alpha_{\mathbf{M}}$ and $\mathbf{c_M}$ agree at ν_0 up through the second derivative. Therefore again since the radius of curvature depends only on the first two derivatives, the radii of curvature of $\alpha_{\mathbf{M}}$ and $\mathbf{c_M}$ are the same. Since the radius of curvature does not depend on a particular parametrization, by Lemma 8.3.3 the common radius of curvature at the point ν_0 is $r_A + r_B$. Since ν_0 was arbitrary, the radius of curvature at each point of $\alpha_{\mathbf{M}}$ is equal to the sum of the radii of curvature of the original two curves. □

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Lemma 8.3.5 *The radius of normal curvature of the Minkowski difference of two disjoint convex bodies in R^3 is equal to the sum of the radii of normal curvature of the original two bodies.*

Proof. Let A and B denote the bodies. Parametrize the boundaries $\mathbf{s_A}$ and $\mathbf{s_B}$ by the angles (θ, ϕ) of the normal vector. Since $\mathbf{s_A}$ and $\mathbf{s_B}$ are parametrized by the normal vector, the tangent planes for a particular value of (θ, ϕ) are parallel. Identify parallel directions in the parallel planes. Let ζ represent a direction in the tangent planes. Then the normal sections in the direction ζ through each tangent plane are convex curves in parallel planes. Therefore by Lemma 8.3.4 above the radius of normal curvature of the Minkowski difference at the point (θ, ϕ) in the direction ζ is equal to the sum of the radii of curvature of $\mathbf{s_A}$ and $\mathbf{s_B}$ at the point (θ, ϕ) in the direction ζ. □

Conjecture 8.3.6 *The mean principle radii of curvature of the Minkowski difference of two disjoint strictly convex bodies in R^3 is equal to the sum of the mean principle radii of curvature of the original two bodies.*

If the conjecture is true then it would follow that the total absolute mean curvature over the cap of the Minkowski difference is equal to the total absolute mean curvature over the caps of the original surfaces.

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A somewhat different approach would have to be taken to equate the total mean curvatures over the envelopes because we can't assume that the principle radii of curvature on the envelopes are finite.

Theorem 8.3.7 *Let A and B be two disjoint compact strictly convex bodies in R^3. Let (x, y, z) denote a point in R^3. Place the z-axis through a shortest line segment between A and B. Let $\mathbf{n_A}(\theta, \phi)$ denote the upward and $\mathbf{n_B}(\theta, \phi)$ denote the downward unit normal vectors of $\mathbf{s_A}$ and $\mathbf{s_B}$ respectively. Parametrize the respective boundaries s_A and s_B by direction (θ, ϕ) of the outward and inward normals respectively where ϕ is the angle of the normal vector with the z-axis and θ is the angle with the x-axis of the projection onto the xy-plane of the upward normal vector. For a fixed θ let $\phi(\theta)$ represent the angle with the z-axis of the normal vector of the separating double tangent plane. Then for $0 \leq \theta < 2\pi$ and $0 \leq t \leq 1$ the envelope of the original surfaces is given by*

$$(\theta, t) \rightarrow s_A(\theta, \phi(\theta)) + t\big(s_B(\theta, \phi(\theta)) - s_A(\theta, \phi(\theta))\big)$$

and the envelope of the Minkowski difference and zero is given by

$$(\theta, t) \rightarrow t\big(s_B(\theta, \phi(\theta)) - s_A(\theta, \phi(\theta))\big).$$

Proof. Let $\phi(\theta)$ denote the function given by Theorem 8.2.2 expressing ϕ as a function of θ at points of separating double tangency. Thus $s_B(\theta, \phi(\theta))$ and $s_A(\theta, \phi(\theta))$ are corresponding points of separating double tangency. Thus the

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envelope for the original two surfaces is

$$(1-t)s_A(\theta, \phi(\theta)) + ts_B(\theta, \phi(\theta))$$

$$= s_A(\theta, \phi(\theta)) + t\big(s_B(\theta, \phi(\theta)) - s_A(\theta, \phi(\theta))\big).$$

Also by Theorem 8.3.1 the surface of the Minkowski difference is given by

$$s_M(\theta, \phi) = s_B(\theta, \phi) - s_A(\theta, \phi).$$

Because of the linearity of derivatives and dot products, the normal vector $n(\theta, \phi)$ of the original surfaces will be orthogonal to both coordinate curves of the difference and thus normal to the difference. Thus the separating double support planes for the Minkowski difference and the origin must satisfy

$$n(\theta, \phi) \cdot s_M(\theta, \phi) = 0.$$

Then replacing s_M by $s_B - s_A$ yields the equation for $\phi(\theta)$ given by Theorem 8.2.2. Thus the separating double tangent points for the Minkowski difference and zero are $s_B(\theta, \phi(\theta)) - s_A(\theta, \phi(\theta))$ and zero. Thus the envelope for the Minkowski difference and zero is

$$(\theta, t) \to t\big(s_B(\theta, \phi(\theta)) - s_A(\theta, \phi(\theta))\big).$$

□

Remark 8.3.8 *As is apparent from the formulas, the envelopes of Theorem 8.3.7 are unions of straight lines. For fixed θ the equation of the envelope*

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reduces to the equation of a straight line. Such surfaces are called **ruled surfaces**. *The straight lines are called the* **rulings** *of the surface. See for example Do Carmo (1976, 188-197) especially the discussion on pages 195 to 197 of the envelope of the family of planes tangent to a surface along a curve of the surface.*

8.4 Stokes' Theorem Approach

A third approach that holds some promise is a Stokes' Theorem approach. The set of planes separating two disjoint compact convex bodies A and B is bounded by the set of separating double support planes and also by the single support planes of the caps of A and B. Thus by a general Stokes' Theorem we can integrate over these bounding planes rather than the separating planes. Hopefully the integral over these bounding planes can then be equated with a mean curvature integral over the points.

Alternatively one could possibly identify the separating planes with points bounded by the envelope and caps and then apply Stokes' Theorem to equate the integral over these points with an integral over the boundary. Hopefully this integral over the boundary would turn out to be the total absolute mean curvature.

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A difficulty in taking a Stokes' Theorem approach lies in finding an expeditious way of identifying planes with points.

8.5 Approximation by Polyhedra

A fourth possible approach to the general smooth case would be to approximate a pair of smooth convex bodies by polyhedra and taking the limit as the polyhedral pair approximates the smooth pair.

We tried this approach numerically for the measure of planes hitting a sphere. We parametrized the sphere by direction (θ, ϕ) of the normal vector. The measure of planes hitting a polyhedron with vertices at $(\theta/n, \phi/n)$ was numerically fairly close to the measure of planes hitting a sphere.

A difficulty in taking the polyhedra approximation approach lies in showing that the limit of the wedge functions is mean curvature.

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CHAPTER 9

PLANES SEPARATING COAXIAL SMOOTH COMPACT SURFACES OF REVOLUTION

9.1 Introduction

In this chapter we will compute the measure of planes separating a pair of disjoint compact convex bodies A and B (with non-empty interior) whose boundaries are coaxial smooth surfaces of revolution in R^3 with parametriza-

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tions of a certain form. We will show that the measure of planes separating the two convex bodies is the integral of the absolute value of the mean curvature over the portion of the envelope of separating double tangent planes that is bounded by the two bodies minus the integral of the absolute value of the mean curvature over portions of the boundaries of the two bodies that are bounded by the envelope.

Part of the proof given below involves showing that the set of tangent planes meet in a point. Thus the results of the Chapter 7 may be applied. However here we will prove the result for this special case independently.

9.2 The Cylindrical Parametrization

Let (x, y, z) denote rectangular coordinates in R^3 and let (θ, ρ) denote polar coordinates in R^2. The boundaries of the convex bodies A and B will be parametrized in R^3 as the graphs of the height functions of the polar coordinates.

Definition 9.2.1 *The* **cylindrical parametrization** *of the boundaries of a pair of disjoint compact convex bodies A and B whose boundaries are coaxial smooth surfaces of revolution is defined as follows. The boundary of the lower*

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body A will have a parametrization $\mathbf{s_A} : [0, 2\pi) \times [0, \alpha] \to R^3$ of the form

$$\mathbf{s_A}(\theta, \rho) = (\rho \cos\theta, \rho \sin\theta, f_A(\rho)) \tag{9.2.2}$$

where $0 < \alpha < \infty$ and $f_A : [0, \alpha] \to R$ is twice differentiable, strictly concave (i.e. $f''(\rho) < 0$ for all ρ), and satisfies $f_A(0) = \lim_{\rho \to 0^-} f_A'(\rho) = 0$ and $\lim_{\rho \to \alpha^-} f_A'(\rho) = -\infty$.

Similarly the boundary of upper body B will have a parametrization $\mathbf{s_B}$: $[0, 2\pi) \times [0, \beta] \to R^3$ of the form

$$\mathbf{s_B}(\theta, \rho) = (-\rho \cos\theta, -\rho \sin\theta, c - f_B(\rho)) \tag{9.2.3}$$

where $0 < \beta < \infty$, $0 < c < \infty$, and $f_B : [0, \beta] \to R$ is twice differentiable, strictly concave, and satisfies $f_B(0) = \lim_{\rho \to 0^-} f_B'(\rho) = 0$ and $\lim_{\rho \to \beta^-} f_B'(\rho) = -\infty$.

Remark 9.2.4 *Thus f_A, f_A', f_B, and f_B' are all nonpositive on their domains and strictly decreasing.*

Remark 9.2.5 *Thus A is contained in the half-space $\{(x, y, z) \in R^3 : z \leq 0\}$ because the z-coordinate of points on the boundary of A is $f_A \leq 0$. Likewise B is contained in the half-space $\{(x, y, z) \in R^3 : z \geq c\}$ because the z-coordinate of points on the boundary of B is $c - f_B \geq c$. Furthermore c is a separation parameter representing the distance between A and B.*

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Remark 9.2.6 *Furthermore* $\mathbf{s_A}$ *is bijective everywhere except at points where* $\rho = 0$ *and* $\mathbf{s_A}^{-1}(0, 0, 0) = \{(\theta, \rho) : \rho = 0\}$. *Likewise* $\mathbf{s_B}$ *is bijective everywhere except at points where* $\rho = 0$ *and* $\mathbf{s_B}^{-1}(0, 0, 0) = \{(\theta, \rho) : \rho = 0\}$.

Lemma 9.2.7 *If* A *and* B *are two disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with cylindrical parametrization then* A *contains the set*

$$\{(x, y, z) \in R^3 : \sqrt{x^2 + y^2} \leq \alpha \text{ and } f_A(\alpha) \leq z \leq f_A(\sqrt{x^2 + y^2})\}$$

and B *contains the set*

$$\{(x, y, z) \in R^3 : \sqrt{x^2 + y^2} \leq \beta \text{ and } c - f_B(\sqrt{x^2 + y^2}) \leq z \leq c - f_B(\beta))\}.$$

Proof: First note that the x-coordinate and the y-coordinate in the cylindrical parametrization are $\rho\cos\theta$ and $\rho\sin\theta$ respectively. Thus $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Solving for ρ yields $\rho = \sqrt{x^2 + y^2}$. Thus the constraint $0 \leq \rho \leq \alpha$ becomes $0 \leq \sqrt{x^2 + y^2} \leq \alpha$.

For $0 \leq \rho \leq \alpha$ the point $(\rho\cos\theta, \rho\sin\theta, f_A(\alpha)) \in A$ because A is convex and because the point $(\rho\cos\theta, \rho\sin\theta, f_A(\alpha))$ is on the line segment connecting the boundary points $(\alpha\cos\theta, \alpha\sin\theta, f_A(\alpha))$ and $(\alpha\cos(\theta + \pi), \alpha\sin(\theta + \pi), f_A(\alpha))$. Thus for z satisfying $f_A(\alpha) \leq z \leq f_A(\rho)$ the point $(\rho\cos\theta, \rho\sin\theta, z) \in A$ because it is on the line segment connecting $(\rho\cos\theta, \rho\sin\theta, f_A(\alpha))$ and

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$(\rho\cos\theta, \rho\sin\theta, f_A(\rho))$. Substituting $\sqrt{x^2+y^2}$ for ρ then implies that A contains

$$\{(x,y,z) \in R^3 : \sqrt{x^2+y^2} \leq \alpha \text{ and } f_A(\alpha) \leq z \leq f_A(\sqrt{x^2+y^2})\}.$$

By a similar argument B contains the set

$$\{(x,y,z) \in R^3 : \sqrt{x^2+y^2} \leq \beta \text{ and } c - f_B(\sqrt{x^2+y^2}) \leq z \leq c - f_B(\beta))\}.$$

□

9.3 The Set of Separating Double Tangent Planes

Lemma 9.3.1 *If A and B are disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with a cylindrical parametrization then a necessary and sufficient set of conditions for points (θ_A, ρ_A) on the boundary of A and (θ_B, ρ_B) on the boundary of B to be points of tangency for the same plane are*

(9.3.2) $$(\theta_A, f'_A(\rho_A)) = (\theta_B, f'_B(\rho_B))$$

(9.3.3) $$f'_A(\rho_A) = f'_B(\rho_B) = \frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}.$$

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Proof: Applying standard calculus and linear algebra techniques to equations 9.2.2 and 9.2.3 above we compute first the tangent vectors to the coordinate curves

$$\mathbf{s_{A\theta}} = (-\rho_A \sin\theta_A, \rho_A \cos\theta_A, 0)$$

$$\mathbf{s_{A\rho}} = (\cos\theta_A, \sin\theta_A, f'_A(\rho_A))$$

$$\mathbf{s_{B\theta}} = (\rho_B \sin\theta_B, -\rho_B \cos\theta_B, 0)$$

$$\mathbf{s_{B\rho}} = (-\cos\theta_B, -\sin\theta_B, -f'_A(\rho_A))$$

and then the upward unit normal vectors

(9.3.4) $$\mathbf{n_A}(\theta_A, \rho_A) = (-f'_A(\rho_A)\cos\theta_A, -f'_A(\rho_A)\sin\theta_A, 1)/\sqrt{1 + (f'_A(\rho_A))^2}$$

(9.3.5) $$\mathbf{n_B}(\theta_B, \rho_B) = (-f'_B(\rho_B)\cos\theta_B, -f'_B(\rho_B)\sin\theta_B, 1)/\sqrt{1 + (f'_B(\rho_B))^2}$$

at points (θ_A, ρ_A) and (θ_B, ρ_B) of the boundaries of A and B respectively.

Note that n_A is an outward normal to surface A and that n_B is an inward normal to surface B. Thus we have chosen orientations for the two surfaces so that for a separating double tangent plane the two unit normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ must be the same. Then solving $\mathbf{n_A} = \mathbf{n_B}$ yields the necessary condition

$$(\theta_A, f'_A(\rho_A)) = (\theta_B, f'_B(\rho_B)).$$

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Next we compute the dot products

$$\mathbf{n_A} \cdot \mathbf{s_A} = (-\rho_A f'_A(\rho_A) + f_A(\rho_A))/\sqrt{1 + f'_A(\rho_A)^2}$$

$$\mathbf{n_B} \cdot \mathbf{s_B} = (\rho_B f'_B(\rho_B) + c - f_B(\rho_B))/\sqrt{1 + f'_B(\rho_B)^2}.$$

Then solving the equation $\mathbf{n_A} \cdot \mathbf{s_A} = \mathbf{n_B} \cdot \mathbf{s_B}$ yields a second condition

$$f'_A(\rho_A) = f'_B(\rho_B) = \frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}$$

necessary for a pair of points on the surfaces to have a tangent plane in common. From basic linear algebra the two necessary conditions are also sufficient.

□

Lemma 9.3.6 *If A and B are disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with a cylindrical parametrization then the equation*

$$f'_A(\rho_A) = f'_B(\rho_B) = \frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}$$

has a unique solution (ρ_{A0}, ρ_{B0}).

Proof: Restricting $\mathbf{s_A}$ and $\mathbf{s_B}$ to the xz-plane by taking $\theta_A = \theta_B = 0$ yields two planar curves

$$\rho_A \to (\rho_A, 0, f_A(\rho_A)) \quad \text{and}$$

$$\rho_B \to (-\rho_B, 0, c - f_B(\rho_B)).$$

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Then using basic planar analytic geometry the expression

$$\frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}$$

which appears in equation 9.3.3 above is just the slope of the line through pairs of points $(\rho_A, 0, f_A(\rho_A))$ and $(-\rho_A, 0, c - f_B(\rho_B))$ on the two curves and $f'_A(\rho_A)$ and $f'_B(\rho_B)$ are the slopes of the tangent lines to the curves. Thus the condition

$$f'_A(\rho_A) = f'_B(\rho_B) = \frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}$$

is equivalent to ρ_A and ρ_B being points of tangency to a separating double tangent line.

Since f_A is strictly concave then f'_A is strictly decreasing and therefore has a strictly decreasing inverse ${f'_A}^{-1} : (-\infty, 0] \to [0, \alpha)$ mapping a slope m to a point on the x-axis ρ. Furthermore the strict concavity of f_A and the fact that $f'_A(0) = 0$ implies that ${f'_A}^{-1}(-\infty, 0) = (0, \alpha)$.

Likewise since f_B is strictly concave then f'_B is strictly decreasing and therefore has a strictly decreasing inverse ${f'_B}^{-1} : (-\infty, 0] \to [0, \beta)$ mapping a slope m to a point on the x-axis ρ and ${f'_B}^{-1}(-\infty, 0) = (0, \beta)$.

Letting $m = f'_A(\rho_A) = f'_B(\rho_B)$ the condition 9.3.3 is equivalent to the condition

$$m = \frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}.$$

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Thus m must satisfy

$$m = \frac{f_A(f_A'^{-1}(m)) + f_B(f_B'^{-1}(m)) - c}{f_A'^{-1}(m) + f_B'^{-1}(m)}.$$

Letting $M : R \to R$ be defined by

$$M(m) = m - \frac{f_A(f_A'^{-1}(m)) + f_B(f_B'^{-1}(m)) - c}{f_A'^{-1}(m) + f_B'^{-1}(m)}.$$

the above condition is equivalent to $M = 0$. Note that since $f_A'^{-1}$ and $f_B'^{-1}$ are continuous and positive on $(-\infty, 0)$ and since f_A and f_B are continuous $[0, \alpha]$ and $[0, \beta]$ respectively then M is continuous on $(-\infty, 0)$. Furthermore

$$\lim_{m \to -\infty} M = \lim_{m \to -\infty} m - \frac{f_A(\lim_{m \to -\infty} f_A'^{-1}(m)) + f_B(\lim_{m \to -\infty} f_B'^{-1}(m)) - c}{\lim_{m \to -\infty} f_A'^{-1}(m) + \lim_{m \to -\infty} f_B'^{-1}(m)}$$

$$= -\infty - \frac{f_A(\alpha) + f_B(\beta) - c}{\alpha + \beta} = -\infty$$

and

$$\lim_{m \to 0^-} M = \lim_{m \to 0^-} m - \frac{f_A(\lim_{m \to 0^-} f_A'^{-1}(m)) + f_B(\lim_{m \to 0^-} f_B'^{-1}(m)) - c}{\lim_{m \to 0^-} f_A'^{-1}(m) + \lim_{m \to 0^-} f_B'^{-1}(m)}$$

$$= 0 - \frac{f_A(0) + f_B(0) - c}{0^+} = \frac{c}{0^+} = \infty.$$

Thus the $\lim_{m \to -\infty} M = -\infty$ and the $\lim_{m \to 0^-} M = \infty$ Therefore the Intermediate Value Theorem guarantees that $M = 0$ for some value m_0 in $(-\infty, 0)$. Thus there is a tangent line corresponding to the slope m_0.

To prove uniqueness suppose that in $(-\infty, 0)$ there are two solutions m_0 and m_1. Without loss of generality assume that $m_0 < m_1$. The equation of

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the tangent line with slope m_0 is

$$z(\rho) = f_A(f_A'^{-1}(m_0)) + m_0(\rho - f_A'^{-1}(m_0)).$$

It is a property of convex bodies that the body is contained in one of the two closed half-planes determined by a support line. Taking $(0, 0, f_A(\alpha))$ as a test point implies that the lower body is contained in the half-plane

$$z \leq f_A(f_A'^{-1}(m_0)) + m_0(\rho - f_A'^{-1}(m_0)).$$

But since both f_A and f_A' are strictly decreasing then $m_0 < m_1$ implies $f_A'^{-1}(m_1) < f_A'^{-1}(m_0)$ which implies

$$f_A(f_A'^{-1}(m_1)) > f_A(f_A'^{-1}(m_0))$$

$$= f_A(f_A'^{-1}(m_0)) + m_0(f_A'^{-1}(m_0) - f_A'^{-1}(m_0))$$

$$> f_A(f_A'^{-1}(m_0)) + m_0(f_A'^{-1}(m_1) - f_A'^{-1}(m_0)).$$

Thus the point $(f_A'^{-1}(m_1), 0, f_A(f_A'^{-1}(m_1)))$ is a point on the boundary of the lower body A that is contained in the upper half plane

$$z > f_A(f_A'^{-1}(m_0)) + m_0(\rho - f_A'^{-1}(m_0)).$$

This is a contradiction. Therefore there is a unique separating double tangent slope m_0 in $(-\infty, 0)$. Thus $\rho_{A0} = f_A'^{-1}(m_0)$ and $\rho_{B0} = f_B'^{-1}(m_0)$. This proves the lemma for the case where $\theta = 0$. The lemma is true for arbitrary θ by rotational symmetry. □

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Corollary 9.3.7 *If A and B are disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with a cylindrical parametrization then the locus of the set of separating double tangent points on A is a simple closed curve given by the formula*

$$\theta \to (\rho_{A0}\cos\theta, \rho_{A0}\sin\theta, f_A(\rho_{A0}))$$

and the locus of the set of separating double tangent points on B is a simple closed curve given by the formula

$$\theta \to (-\rho_{B0}\cos\theta, -\rho_{B0}\sin\theta, c - f_B(\rho_{B0})).$$

Corollary 9.3.8 *If A and B are disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with a cylindrical parametrization then the set of separating double tangent planes is the set of all planes which may be represented by equations of the form*

$$\mathbf{n_A}(\theta, \rho_{A0}) \cdot (x, y, z) = \mathbf{n_A}(\theta, \rho_{A0}) \cdot \mathbf{s_A}(\theta, \rho_{A0}) \text{ or equivalently}$$

$$\mathbf{n_B}(\theta, \rho_{B0}) \cdot (x, y, z) = \mathbf{n_B}(\theta, \rho_{B0}) \cdot \mathbf{s_B}(\theta, \rho_{B0}).$$

Theorem 9.3.9 *If A and B are disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with a cylindrical parametrization then the set of separating double tangent planes intersect in the point* $(0, 0, -\rho_{A0}f'_A(\rho_{A0}) + f_A(\rho_{A0}))$.

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Proof: Let $(x, y, z) \in R^3$. We compute the dot products

$$\mathbf{n_A}(\theta, \rho_{A0}) \cdot (x, y, z) = \frac{-(x\cos\theta + y\sin\theta)f'_A(\rho_{A0}) + z}{\sqrt{1 + f'_A(\rho_{A0})^2}} \text{ and}$$

$$\mathbf{n_A}(\theta, \rho_{A0}) \cdot \mathbf{s_A}(\theta, \rho_{A0}) = \frac{-\rho_{A0} f'_A(\rho_{A0}) + f_A(\rho_{A0})}{\sqrt{1 + f'_A(\rho_{A0})^2}}.$$

Solving for z then yields

$$z = (x\cos\theta + y\sin\theta - \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0}).$$

Then computing z for $\theta = 0, \pi, \pi/2, 3\pi/2$ gives the values listed in Table 9.3.

Table 9.1: Some Separating Planes for Surfaces of Revolution

Angle	Equation of Plane
0	$z = (x - \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0})$
π	$z = -(x + \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0})$
$\pi/2$	$z = (y - \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0})$
$3\pi/2$	$z = -(y + \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0})$

Then setting the first two expressions for z equal gives the equation

$$(x - \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0}) = -(x + \rho_{A0})f'_A(\rho_{A0}) + f_A(\rho_{A0}).$$

Then solving for x implies $x = 0$. Then setting the above two expressions for z equal and solving for y implies $y = 0$.

Next we replace x and y by 0 in the expression for z to give $z = -\rho_{A0} f'_A(\rho_{A0}) + f_A(\rho_{A0})$. Thus these 4 separating double tangent planes meet in the point

$$p_0 = (0, 0, -\rho_{A0} f'_A(\rho_{A0}) + f_A(\rho_{A0})).$$

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Thus the set of all separating double tangent planes meet in at most the single point p_0.

Finally we compute

$$\mathbf{n_A}(\theta, \rho_{A0}) \cdot p_0 = \frac{-\rho_{A0} f'_A(\rho_{A0}) + f_A(\rho_{A0})}{\sqrt{1 + f'_A(\rho_{A0})^2}} = \mathbf{n_A}(\theta, \rho_{A0}) \cdot \mathbf{s_A}(\theta, \rho_{A0})$$

which shows that the point p_0 lies in all of the separating double tangent planes. Thus the set of separating double tangent planes meet in the point p_0. □

9.4 The Envelope of Separating Double Tangent Planes

Theorem 9.4.1 *If A and B are disjoint compact convex bodies whose boundaries are smooth coaxial surfaces of revolution with a cylindrical parametrization then there is an envelope of separating double tangent planes which may be parametrized by* $\mathbf{s} : [0, 2\pi) \times (-\infty, \infty) \to R^3$ *given by* $\mathbf{s}(\theta, t) =$

$$(((1-t)\rho_{A0} - t\rho_{B0})\cos\theta, ((1-t)\rho_{A0} - t\rho_{B0})\sin\theta, (1-t)f_A(\rho_{A0}) + t(c - f_B(\rho_{B0})))$$

where (ρ_{A0}, ρ_{B0}) *is the unique solution of*

$$f'_A(\rho_A) = f'_B(\rho_B) = \frac{f_A(\rho_A) + f_B(\rho_B) - c}{\rho_A + \rho_B}.$$

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Proof: According to Lemmas 9.3.3 and 9.3.6 above there exist unique numbers ρ_{A0} and ρ_{B0} such that for all values of θ the points (θ, ρ_{A0}) on the boundary of A and (θ, ρ_{B0}) on the boundary of B are corresponding points of double tangency. Specifically

$$(9.4.2) \qquad f'_A(\rho_{A0}) = f'_B(\rho_{B0}) = \frac{f_A(\rho_{A0}) + f_B(\rho_{B0}) - c}{\rho_{A0} + \rho_{B0}}.$$

Now let S be the image of the map $\mathbf{s} : [0, 2\pi) \times (-\infty, \infty) \to R^3$ given by $\mathbf{s}(\theta, t) =$

$$(((1-t)\rho_{A0} - t\rho_{B0})\cos\theta, ((1-t)\rho_{A0} - t\rho_{B0})\sin\theta, (1-t)f_A(\rho_{A0}) + t(c - f_B(\rho_{B0}))).$$

We check that the set of tangent planes of S coincides with the set of separating double tangent planes of the boundaries of A and B as follows. The tangent vectors to the coordinate curves of S are

$$\mathbf{s}_\theta(\theta, t) = (((t-1)\rho_{A0} + t\rho_{B0})\sin\theta, ((1-t)\rho_{A0} - t\rho_{B0})\cos\theta, 0)$$

$$\mathbf{s}_t(\theta, t) = (-(\rho_{A0} + \rho_{B0})\cos\theta, -(\rho_{A0} + \rho_{B0})\sin\theta, c - f_A(\rho_{A0}) - f_B(\rho_{B0})).$$

Thus the normal vector is

$$\mathbf{n}(\theta, t) = \frac{\left(\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0} + \rho_{B0}}\cos\theta, \frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0} + \rho_{B0}}\sin\theta, 1\right)}{\sqrt{1 + \left(\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0} + \rho_{B0}}\right)^2}}.$$

Thus, using equation 9.4.2 above,

$$\mathbf{n}(\theta, t) = (-f'_A(\rho_{A0})\cos\theta, -f'_A(\rho_{A0})\sin\theta, 1)/\sqrt{1 + (f'_A(\rho_{A0}))^2}\mathbf{n_A}(\theta, \rho_{A0}) \text{ and}$$

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$$\mathbf{n}(\theta, t) = (-f'_B(\rho_{B0})\cos\theta, -f'_B(\rho_{B0})\sin\theta, 1)/\sqrt{1 + (f'_B(\rho_{B0}))^2}\mathbf{n_B}(\theta, \rho_{B0}).$$

Thus the set of normal directions of S is exactly the set of normal directions of the separating double tangent planes to S_A and S_B.

Furthermore

$$\mathbf{n}\cdot\mathbf{s} = \frac{((1-t)\rho_{A0} - t\rho_{B0})\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0}+\rho_{B0}} + (1-t)f_A(\rho_{A0}) + t(c - f_B(\rho_{B0}))}{\sqrt{1 + \left(\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0}+\rho_{B0}}\right)^2}}$$

$$= (1-t)\left(f_A(\rho_{A0}) + \rho_{A0}\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0}+\rho_{B0}}\right)/\sqrt{1 + \left(\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0}+\rho_{B0}}\right)^2}$$

$$+t\left(c - f_B(\rho_{B0}) + \rho_{B0}\frac{f_A(\rho_{A0}) + f_B(\rho_{B0} - c)}{\rho_{A0}+\rho_{B0}}\right)/\sqrt{1 + \left(\frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{\rho_{A0}+\rho_{B0}}\right)^2}.$$

Thus, again using equation 9.4.2 above,

$$\mathbf{n}\cdot\mathbf{s} = (1-t)\left(-\rho_{A0}f'_A(\rho_{A0}) + f_A(\rho_{A0}\right)/\sqrt{1 + (f'_A(\rho_{A0})^2}$$

$$+t\left(\rho_{B0}f'_B(\rho_{B0}) + c - f_B(\rho_{B0}\right)/\sqrt{1 + (f'_B(\rho_{B0})^2}$$

$$= (1-t)\mathbf{n}_A(\theta, \rho_{A0})\cdot\mathbf{s}_A(\theta, \rho_{A0}) + t\mathbf{n}_B(\theta, \rho_{B0})\cdot\mathbf{s}_B(\theta, \rho_{B0}).$$

Thus, since $\mathbf{n}_A = \mathbf{n}_B$ at points of double tangency we have

$$\mathbf{n}\cdot\mathbf{s} = \mathbf{n}_A(\theta, \rho_{A0})\cdot\mathbf{s}_A(\theta, \rho_{A0}) = \mathbf{n}_B(\theta, \rho_{B0})\cdot\mathbf{s}_B(\theta, \rho_{B0}).$$

□

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9.5 Direct Computation of Measure of Separating Planes

Theorem 9.5.1 *If A and B are two disjoint compact smooth convex bodies in R^3 whose boundaries are smooth coaxial surfaces of revolution and have a polar parametrization, then the measure of planes separating the A and B is*

$$\pi\left[c+\int_0^{\rho_{A0}}\frac{\rho f_A''(\rho)}{1+(f_A'(\rho))^2}d\rho+\int_0^{\rho_{B0}}\frac{\rho f_B''(\rho)}{1+(f_B'(\rho))^2}d\rho\right].$$

Proof: Recall the equations 9.3.4 and 9.3.5

$$\mathbf{n_A}(\theta,\rho)=(-f_A'(\rho)\cos\theta,-f_A'(\rho)\sin\theta,1)/\sqrt{1+(f_A'(\rho))^2}$$

$$\mathbf{n_B}(\theta,\rho)=(-f_B'(\rho)\cos\theta,-f_B'(\rho)\sin\theta,1)/\sqrt{1+(f_B'(\rho))^2}$$

of the equations of the normal vectors to S_A and S_B respectively. Projecting n_A and n_B onto the xy-plane and normalizing yields vectors

$$proj_{nA}=proj_{nB}=(\cos\theta,\sin\theta,0).$$

Thus the angle of the x-axis with $proj_{nA}$ is θ. Likewise the angle of the x-axis with $proj_{nB}$ is θ.

Let (x,y,z) denote a point in R^3. One can coordinatize almost all planes in R^3 with coordinates (θ,ϕ,z) where $\phi\in(0,\pi)$ denotes the angle of a normal to the plane with the z-axis and $\theta\in(0,2\pi)$ denotes the angle of the projection

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onto the xy-plane of a normal to the plane with the x-axis and by abuse of notation z denotes the point where the plane hits the z-axis. In these coordinates the measure of the set of planes in R^3 has density $\cos\phi \sin phi\ d\phi\ d\theta\ dz$. See Ambartzumian (1990, 53).

Find the z-coordinate of the point where the envelope intersects the z-axis as follows. Solving $(1-t)\rho_{A0} - t\rho_{B0} = 0$ yields $t = \frac{\rho_{A0}}{\rho_{A0}+\rho_{B0}}$ which implies that the z-coordinate is

$$c_0 = \frac{\rho_{B0} f_A(\rho_{A0})}{\rho_{A0}+\rho_{B0}} + \frac{\rho_{A0}(c - f_B(\rho_{B0}))}{\rho_{A0}+\rho_{B0}}.$$

For $z_0 < c_0$ find the ρ-coordinates of the separating planes by solving $\mathbf{n_A} \cdot \mathbf{s_A} = \mathbf{n_A} \cdot (0.0.z_0)$. This simplifies to an equation

$$f'_A(\rho_A) = \frac{f_A(\rho_A) - z_0}{\rho_A}.$$

We will now show that this equation has a solution. Define a function F : $(0.\alpha) \to R$ by

$$F(\rho_A) = f'_A(\rho_A) - \frac{f_A(\rho_A) - z_0}{\rho_A}.$$

Note that F inherits continuity from f_A and that $\lim_{\rho_A \to 0^+} F(\rho_A) = \infty$ and $\lim_{\rho_A \to \alpha^-} F(\rho_A) = -\infty$. Therefore by the Intermediate Value Theorem $F(\rho_z) = 0$ for some $\rho_z \in (0.\alpha)$.

We will now show that the solution ρ_z is unique. Suppose there is a second solution ρ_{z2}. Without loss of generality assume $\rho_z < \rho_{z2}$. Then since f_A is

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concave $f'_A(\rho_z) > f'_A(\rho_{z2})$. Since the only restriction on θ_A and θ_B is that $\theta_A = \theta_B$ we can take $\theta_A = \theta_B = 0$ and consider tangent lines $z = f'_A(\rho_z)\rho + z_0$ and $z = f'_A(\rho_{z2})\rho + z_0$ to the curve $\rho \to (\rho, 0, f_A(\rho))$ in the xz-plane. Since the curve is concave it can be extended to form the boundary of a convex body in R^2. It is a property of convex bodies in R^2 that the body is contained in one of the half-planes with boundary bounded by a tangent line. Taking (0.0.0) as a test point this implies that all points z in the convex body must satisfy $z \leq f'_A(\rho_{z2})\rho + z_0$. But $f_A(\rho_z) = f'_A(\rho_z)\rho_z + z_0 > f'_A(\rho_{z2})\rho_z + z_0$. This is a contradiction. Therefore the solution ρ_z is unique.

Then using elementary trigonometry one can show that the ϕ-coordinate of the tangent plane through a point with ρ-coordinate ρ_z is

$$\phi_z = \cos^{-1}\left(\frac{-f'_A(\rho_z)}{\sqrt{1+(f'_A(\rho_z))^2}}\right).$$

Similarly for $z > c_0$

$$\phi_z = \cos^{-1}\left(\frac{-f'_B(\rho_z)}{\sqrt{1+(f'_B(\rho_z))^2}}\right).$$

Using the symmetries of surfaces of revolution yields the measure of the planes separating A and B as

$$2\int_0^{c_0}\int_0^{\pi}\int_{\phi_z}^{\pi/2}\cos\phi\sin\phi\, d\phi\, d\theta\, dz + 2\int_{c_0}^{c}\int_0^{\pi}\int_{\phi_z}^{\pi/2}\cos\phi\sin\phi\, d\phi\, d\theta\, dz.$$

This simplifies to

$$\pi\left(c - \int_0^{c_0}\frac{1}{1+(f'_A(\rho_z))^2}dz - \int_{c_0}^{c}\frac{1}{1+(f'_B(\rho_z))^2}dz\right).$$

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A change of variables $z = f(\rho) - \rho f'(\rho)$ then yields

$$\pi\left[c + \int_0^{\rho_{A0}} \frac{\rho f_A''(\rho)}{1+(f_A'(\rho))^2}d\rho + \int_0^{\rho_{B0}} \frac{\rho f_B''(\rho)}{1+(f_B'(\rho))^2}d\rho\right].$$

□

9.6 Integral of Mean Curvature

Theorem 9.6.1 *Let A and B be disjoint compact smooth bodies in R^3 whose boundaries are coaxial surfaces of revolution and have a polar parametrization. Then the measure of planes separating A and B is*

(9.6.2)
$$\int_{S_E} |H|dS - \int_{S_C} |H|dS$$

where H is the mean curvature, S_E is the portion of the envelope bounded by the points of tangency, S_C is the union of the two caps, and dS is the measure of points on the surfaces.

Proof: Note that the set of points of tangency to the boundary of A can be parametrized by restricting the ρ-coordinate of $\mathbf{s_A}$ to ρ_{A0} to get a differentiable simple closed curve on the boundary of A given by the formula $\theta \rightarrow (\rho_{A0}\cos\theta, \rho_{A0}\sin\theta, f_A(\rho_{A0}))$ where $\theta \in [0, 2\pi)$. This curve separates the boundary of A into two sets, one with points whose ρ-coordinate are greater than ρ_{A0} and one with points whose ρ -coordinates are less than ρ_{A0}. A similar argument can be made for the points of B. Thus S_C is well-defined.

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We compute the integrals of formula 9.6.2 using local coordinates and basic differential geometry. See for example Do Carmo (1976). Let E,F, and G be coefficients of the first fundamental form for the envelope S_E. Then in local coordinates on the envelope

$$|H| = \frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{2\sqrt{EG - F^2}}$$

and thus

$$\int_{S_E} |H| dS = \int_0^1 \int_0^{2\pi} |H| \sqrt{EG - F^2} d\theta dt$$

$$(9.6.3) \quad = \int_0^1 \int_0^{2\pi} \frac{c - f_A(\rho_{A0}) - f_B(\rho_{B0})}{2} d\theta dt = \pi(c - f_A(\rho_{A0}) - f_B(\rho_{B0})).$$

Now let E.F. and G be coefficients of the first fundamental form on the cap of A. Then in local coordinates on this cap

$$|H| = \frac{-f_1'(\rho)(1 + (f_1'(\rho))^2) - \rho f_A''(\rho)}{2(1 + (f_A'(\rho))^2)\sqrt{EG - F^2}}.$$

Thus the integral of the absolute value of mean curvature over the cap S_{CA} of A is

$$\int_{S_{CA}} |H| dS = \int_0^{\rho_{A0}} \int_0^{2\pi} |H| \sqrt{EG - F^2} d\theta d\rho$$

$$(9.6.4) \qquad = \pi \left[-f_A(\rho_{A0}) - \int_0^{\rho_{A0}} \frac{\rho f_A''(\rho)}{1 + (f_A'(\rho))^2} d\rho \right].$$

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Similarly the integral of the absolute value of mean curvature over the cap S_{CB} of B is

$$\int_{S_{CB}} = \pi\left[-f_B(\rho_{B0}) - \int_0^{\rho_{B0}} \frac{\rho f_B''(\rho)}{1+(f_B'(\rho))^2}d\rho\right]. \tag{9.6.5}$$

Then adding and subtracting equations 9.6.3, 9.6.4, and 9.6.5 above gives

$$\int_{S_E}|H|dS - \int_{S_C}|H|dS = \int_{S_E}|H|dS - \int_{S_{CA}}|H|dS - \int_{S_{CB}}|H|dS$$
$$= \pi\left[c + \int_0^{\rho_{A0}} \frac{\rho f_A''(\rho)}{1+(f_A'(\rho))^2}d\rho + \int_0^{\rho_{B0}} \frac{\rho f_B''(\rho)}{1+(f_B'(\rho))^2}d\rho\right].$$

This is the same as the measure of planes separating A and B by the results of Section 9.5. □

The main reason for the assumption of compactness is to assure the existence of the envelope. For noncompact surfaces of revolution, if the envelope exists, the rotational symmetry of the original pair of convex bodies will force the envelope to be symmetric and thus conical. Thus by the comment following Theorem 7.4.3 the conclusion of Theorem 9.6.1 is true for noncompact surfaces of revolution provided the envelope exists.

9.7 Examples

Example 9.7.1 *Let A and B be a pair of balls whose boundaries are parametrized as above using $f_A(\rho) = \sqrt{r_A^2 - \rho^2} - r_A$, $f_B(\rho) = \sqrt{r_B^2 - \rho^2} - r_B$ and $c = d - r_A - r_B$ where r_A and r_B are the radii of the balls and d is the distance*

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between the centers of the balls. Then a straightforward calculation using the above formula gives the separating measure as

$$\frac{\pi(d - r_A - r_B)^2}{d}.$$

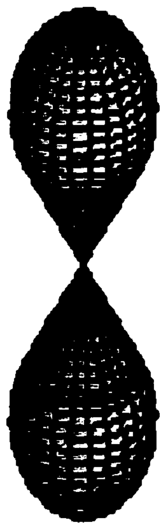

Figure 9.1: Two Spheres and the Envelope

Example 9.7.2 *Let A and B be bounded by a pair of ellipsoids of revolution parametrized as above using $f_A(\rho) = f_B(\rho) = \frac{b}{a}\sqrt{a^2 - \rho^2} - b$ where $a > b$. Then with the help of the Maple mathematical software and the above formulas the separating measure was computed to be*

$$\pi\left\{c + \frac{a^2}{\sqrt{a^2 - b^2}}\tan^{-1}\left(\frac{-2c(2a^2 + bc)\sqrt{a^2 - b^2}}{4a^4 + 4a^2bc - a^2c^2 + 2b^2c^2}\right)\right\}.$$

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Taking the limit as $a \to b^+$ then gives

$$\pi \frac{c^2}{c + 2a}$$

which is the measure of planes separating two spheres of radius a and of distance c apart and agrees with Example 9.7.1 above.

Example 9.7.3 *Let A and B be bounded by a pair of paraboloids of revolution parametrized as above using $f_A : [0, \infty) \to R$ defined by $f_A(\rho) = -a\rho^2$, $f_B : [0, \infty) \to R$ defined by $f_B(v) = -b\rho^2$ where $a > 0$ and $b > 0$. Although the domains of f_A and f_B in this case are infinite the above proofs remain valid and the separating measure can be calculated from the above formulas as*

$$\pi \left[c - \frac{a+b}{4ab} \ln(1 + \frac{4abc}{a+b}) \right].$$

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CHAPTER 10

SEPARATING MEASURE FOR A SPECIFIC FOURTH ORDER SURFACE

10.1 Introduction

The specific smooth pairs of convex bodies we have studied up to this point have been second order surfaces of revolution. In this chapter we will study the measure of planes in R^3 separating a specific pair of fourth order surfaces which doesn't have rotational symmetry but has enough symmetry that the envelope is a cone. The computations were more difficult for this example but the principles remain the same.

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10.2 Mean Curvature Integral

Theorem 10.2.1 *Let $c > 0$. Let A be a body whose boundary is parametrized by $s_A(x, y) = (x, y, -x^4 - y^4)$. Let B be a body whose boundary is parametrized by $s_B(x, y) = (-x, -y, x^4 + y^4 + c)$. Then the integral of absolute mean curvature over the envelope is*

$$\int_0^{(\frac{c}{12})^{1/4}} \frac{32x^2(9x^2 + 4c^2 + 9\sqrt{c/6 - x^4})}{3(c/6 - x^4)^{1/4}(16x^6 + 1 + 16(c/6 - x^4)^{3/2})} dx.$$

Proof. At a point $s_A(x_A, y_A)$ the upward normal to the first surface

$$n_A = \left(\frac{4x_A^3}{\sqrt{1 + 16x_A^6 + 16y_A^6}}, \frac{4y_A^3}{\sqrt{1 + 16x_A^6 + 16y_A^6}}, \frac{1}{\sqrt{1 + 16x_A^6 + 16y_A^6}} \right).$$

At a point $s_B(x_B, y_B)$ the upward normal to the second surface

$$n_B = \left(\frac{4x_B^3}{\sqrt{1 + 16x_B^6 + 16y_B^6}}, \frac{4y_B^3}{\sqrt{1 + 16x_B^6 + 16y_B^6}}, \frac{1}{\sqrt{1 + 16x_B^6 + 16y_B^6}} \right).$$

Setting $n_A = n_B$ yields $x_A = x_B$ and $y_A = y_B$.

By symmetry it suffices to integrate eight times the absolute mean curvature over one eighth of the envelope. Thus we can restrict values of (x, y) to $\{(x, y) : 0 \le x \le y\}$. Solving $n_A \cdot s_A = n_B \cdot s_B$ yields $y = (c/6 - x^4)^{1/4}$. Setting $x = y$ then yields $x = (c/12)^{1/4}$.

Then replacing y with $(c/6 - x^4)^{1/4}$ in the respective formulas for s_A and s_B yields pairs of tangent points $(x, (c/6 - x^4)^{1/4}, -c/6)$ and $(-x, -(c/6 - x^4)^{1/4}, 7c/6)$. The line through corresponding double tangent points may then

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be parametrized

$$t \to ((1-2t)x, (1-2t)(c/6-x^4)^{1/4}, (8t-1)c/6).$$

Thus for $\{(x,t) : 0 \le t \le 1 \quad \text{and} \quad 0 \le x \le (c/12)^{1/4}\}$ the envelope may be parametrized by

$$\text{env}(x,t) = ((1-2t)x, (1-2t)(c/6-x^4)^{1/4}, (8t-1)c/6).$$

Letting H represent the mean curvature of the envelope and dS an element of surface area on the envelope, a long but straightforward computation then yields

$$|H|dS = \frac{4x^2(9x^2+4c^2+9\sqrt{c/6-x^4})}{3(c/6-x^4)^{1/4}(16x^6+1+16(c/6-x^4)^{3/2})}dtdx.$$

Thus the integral of absolute mean curvature over the envelope is

$$\int_0^{(\frac{c}{12})^{1/4}} \int_0^1 \frac{32x^2(9x^2+4c^2+9\sqrt{c/6-x^4})}{3(c/6-x^4)^{1/4}(16x^6+1+16(c/6-x^4)^{3/2})}dtdx$$
$$= \int_0^{(\frac{c}{12})^{1/4}} \frac{32x^2(9x^2+4c^2+9\sqrt{c/6-x^4})}{3(c/6-x^4)^{1/4}(16x^6+1+16(c/6-x^4)^{3/2})}dx.$$

□

Theorem 10.2.2 *Let $c > 0$. Let A be a body whose boundary is parametrized by $s_A(x,y) = (x, y, -x^4-y^4)$. Let B be a body whose boundary is parametrized by $s_B(x,y) = (-x,-y,x^4+y^4+c)$. Then the integral of absolute mean curvature over the caps is*

$$\int_0^{(c/6)^{1/4}} \int_0^{(c/6-x^4)^{1/4}} \frac{48(x^2+16x^2y^6+x^2+16x^6y^2)}{1+16x^6+16y^6}dydx.$$

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Proof. By symmetry it suffices to integrate eight times the absolute mean curvature of the cap of s_A over the first quadrant. Noting that the cap is bounded by the set of points of double tangency and setting $n_A \cdot s_A = n_B \cdot s_B$ yields $x^4 + y^4 = c/6$ Thus the domain of integration will be $\{(x, y) : 0 \le y \le (c/6 - x^4)^{1/4}, 0 \le x \le (c/6)^{1/4}\}$.

Letting H represent the mean curvature of the cap of s_A and dS an element of surface area on the cap of s_A, a straightforward computation then yields

$$|H|dS = \frac{6(x^2 + 16x^2y^6 + x^2 + 16x^6y^2)}{1 + 16x^6 + 16y^6}dydx.$$

Thus the integral of absolute mean curvature over the caps is

$$\int_0^{(c/6)^{1/4}} \int_0^{(c/6-x^4)^{1/4}} \frac{48(x^2 + 16x^2y^6 + x^2 + 16x^6y^2)}{1 + 16x^6 + 16y^6}dydx.$$

□

10.3 Direct Measure

Theorem 10.3.1 *Let $c > 0$. Let A be a body whose boundary is parametrized by $s_A(x, y) = (x, y, -x^4 - y^4)$. Let B be a body whose boundary is parametrized by $s_B(x, y) = (-x, -y, x^4 + y^4 + c)$. Then the measure of planes separating A and B is*

$$\int_0^{2\pi} \int_0^{c/2} \frac{16z^{3/2}}{16z^{3/2} + 3\sqrt{3}(\sin^{4/3}\theta + \cos^{4/3}\theta)^{3/2}}dz\, d\theta.$$

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Proof. Coordinatize almost all planes in R^3 with coordinates (ϕ, θ, z) where ϕ is the angle of the upward normal vector with the positive z-axis, θ is the angle of the projection of the upward normal vector onto the xy-plane with the positive x-axis, and z is the intersection of the plane with the z-axis. Note that the letter z is used here in several different ways but hopefully the meaning will be clear enough from the context. In $\theta z\phi$-coordinates the motion-invariant measure on the set of planes in R^3 is given by Ambartzumian (1990, 53) as $\cos\phi \sin\phi d\phi\ dz\ d\theta$.

Note that for arbitrary (x, y) the point $(0, 0, -x^4 - y^4) \in A$ and the point $(0, 0, x^4 + y^4 + c) \in B$. Thus the z-coordinate of any plane which separates A and B must be between 0 and c. Furthermore, because of symmetry, to get the measure of planes separating A and B the z-variable only needs to be integrated from 0 to $c/2$ and the result multiplied by two.

The set of planes separating A and B are bounded by the set of planes tangent to either A or B. By a straightforward computation, the upward normal vector to s_A is

$$n_A = \left(\frac{4x^3}{\sqrt{1+16x^6+16y^6}}, \frac{4y^3}{\sqrt{1+16x^6+16y^6}}, \frac{1}{\sqrt{1+16x^6+16y^6}}\right).$$

Then solving $n_A \cdot s_A = n_A \cdot (o, o, z)$ for y implies that $y = (z/3 - x^4)^{1/4}$ at a point(x, y) of tangency to A.

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Furthermore at such a point of tangency

$$\cos\theta = \left(\frac{x^3}{\sqrt{x^6+y^6}}, \frac{y^3}{\sqrt{x^6+y^6}}, 0\right)\cdot(1,0,0) = \frac{x^3}{\sqrt{x^6+y^6}} = \frac{x^3}{\sqrt{x^6+(z/3-x^4)^{3/2}}}$$

which implies that $x = \frac{z^{1/4}\cos^{1/3}\theta}{3^{1/4}(\cos^{4/3}\theta+\sin^{4/3}\theta)^{1/4}}$. Furthermore $y = (z/3 - x^4)^{1/4}$

$$= \left(z/3 - \left(\frac{z^{1/4}\cos^{1/3}\theta}{3^{1/4}(\cos^{4/3}\theta+\sin^{4/3}\theta)^{1/4}}\right)^4\right)^{1/4} = \frac{z^{1/4}\sin^{1/3}\theta}{3^{1/4}(\cos^{4/3}\theta+\sin^{4/3}\theta)^{1/4}}.$$

Also at such a point of tangency $\cos\phi = n_A\cdot(0,0,1) = \frac{1}{\sqrt{1+16x^6+16y^6}}$. Thus

$$\cos^2\phi = \frac{1}{1+16x^6+16y^6} = \frac{1}{1+\frac{16z^{3/2}\cos^2\theta}{3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}+\frac{16z^{3/2}\sin^2\theta}{3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}}$$

$$= \frac{3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}{16z^{3/2}+3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}.$$

Thus the measure of planes separating the two surfaces is

$$2\int_0^{2\pi}\int_0^{\frac{c}{2}}\int_0^{\cos^{-1}\sqrt{\frac{3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}{16z^{3/2}+3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}}}\cos\phi\sin\phi d\phi\ dz\ d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{c}{2}} -\cos^2 v\Big|_0^{\cos^{-1}\sqrt{\frac{3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}{16z^{3/2}+3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}}} dz\ d\theta$$

$$= \int_0^{2\pi}\int_0^{c/2}\frac{16z^{3/2}}{16z^{3/2}+3\sqrt{3}(\sin^{4/3}\theta+\cos^{4/3}\theta)^{3/2}}dz\ d\theta.$$

□

10.4 Main Theorem

Theorem 10.4.1 *Let $c > 0$. Let A be a body whose boundary is parametrized by $s_A(x,y) = (x, y, -x^4-y^4)$. Let B be a body whose boundary is parametrized*

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by $s_B(x,y) = (-x, -y, x^4 + y^4 + c)$. *Then the measure of planes separating* A *and* B *is equal to the total absolute mean curvature over the envelope minus the total absolute mean curvature over the caps.*

Proof. Because A and B are symmetric relative to the point $0, 0, c/2)$ this theorem is actually a corollary of Theorem 7.4.3.

Numerical Evidence. To provide numerical evidence for the theorem. the direct integrals and mean curvature integrals were computed numerically with a Fortran computer program using Simpson's Rule. Temple graduate student Jian Jun Xu assisted this author by writing a Fortran program that applied Simpson's Rule to the integral of mean curvature over the caps and which could be easily adapted to the other two integrals. For each integral we experimented with different numbers of nodes and tried to strike a balance between speed and precision. Table 10.5 gives the results for various values of c. the distance between the surfaces. More details are given in Appendix A.

10.5 Figures and Tables

Figure 10.1 below shows the envelope of separating double tangent planes along with the caps. Note that this is an example of a pair of smooth surfaces

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which are not surfaces of revolution but nevertheless has a conical envelope. The envelope shares the four-fold rotational symmetry of the original pair.

Figure 10.2 is a graph of the integrand of the direct integral in terms of its coordinates. The figure clearly has four-fold translational symmetry and also reflective symmetry in the θ variable. This symmetry comes from the sum of even powers of functions of $\sin\theta$ and $\cos\theta$ in the denominator of the integrand. Therefore when we numerically integrated we saved considerable time by integrating the θ variable only from 0 to $\pi/4$ and also multiplying the integrand by 8. See Appendix A.

Figure 10.3 is a graph of the integrand of the mean curvature over the caps. Notice the four-fold rotational symmetry and also the reflective symmetry. We already took advantage of the rotational symmetry in deriving the formula in Theorem 10.2.2. Further time was saved in numerically integrating by also taking advantage of the reflective symmetry and multiplying the integrand by two.

A problem sometimes arose in numerically integrating the mean curvature over the caps because one of the limits of integration was a fourth root of a positive number close to zero but because of rounding it sometimes came out to be the fourth root of a negative number. This problem was avoided by dividing and reducing the integration region as is shown in the shaded areas

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of Figures 10.4 and 10.5. See Appendix A for the actual Fortran computer program.

Table 10.1: Fourth Order Surfaces Compared Numerically

Distance	Separating Measure	Mean Curvature Integral
1	0.728378874232 874	0.728378874232 79
2	2.63631526835390	2.63631526835390
3	5.061597926815 48	5.061597926815 77
4	7.730606887289 56	7.730606887289 43
5	10.53295843036 70	10.53295843036 681
6	13.41665770598 72	13.41665770598 673
7	16.354001011650 8	16.354001011650 34
8	19.32881108893 77	19.32881108893 651
17	46.8261435485 230	46.8261435485 1628
27	77.8847299848 921	77.8847299848 7195
37	109.099436078 202	109.099436078 1591
47	140.3821089281 90	140.3821089281 14
57	171.701492283 835	171.701492283 714
67	203.043337536 226	203.043337536 0398
77	234.400101149 388	234.400101149 1305
87	265.767367386 731	265.767367386 3842

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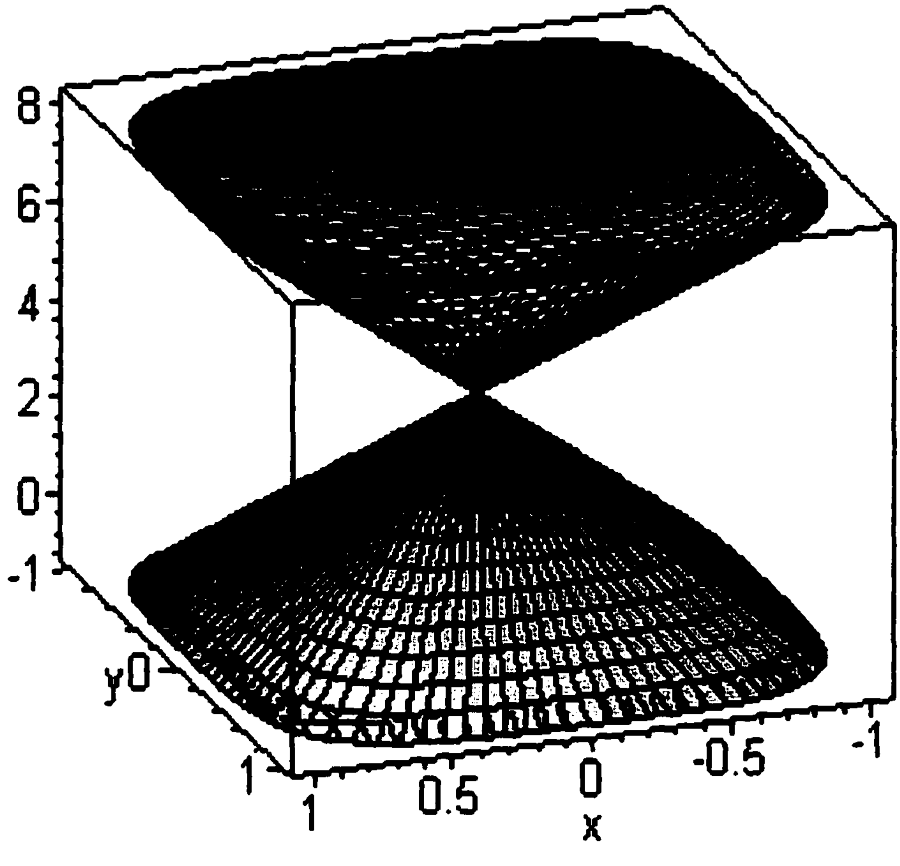

Figure 10.1: Fourth Order Caps With Envelope

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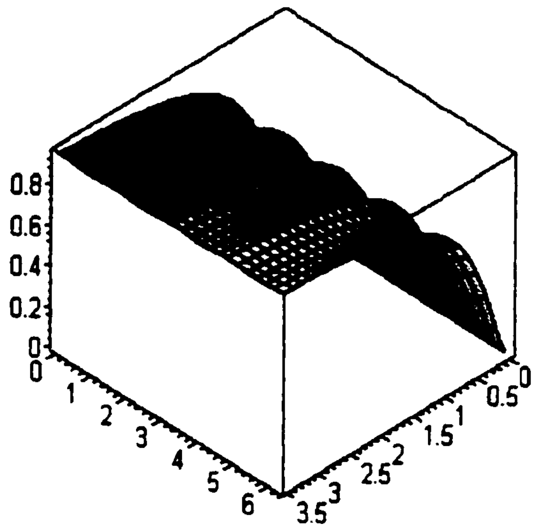

Figure 10.2: Fourth Order Direct Integrand

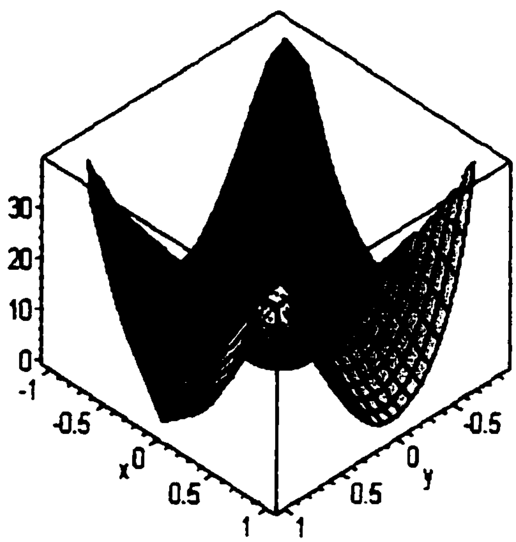

Figure 10.3: Fourth Order Caps Integrand

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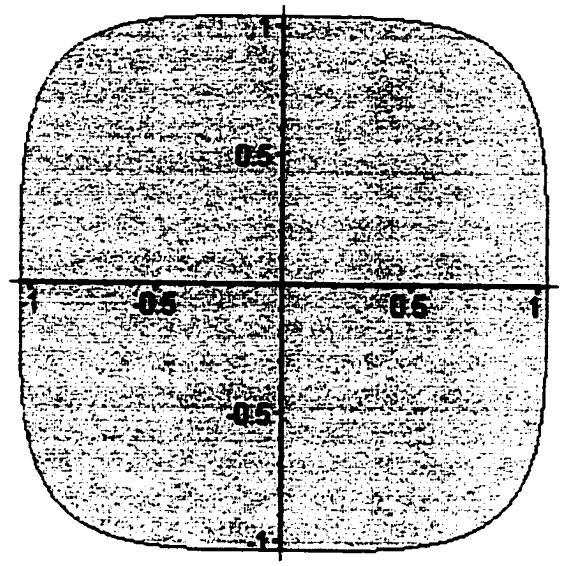

Figure 10.4: Fourth Order Cap Original Integration Region

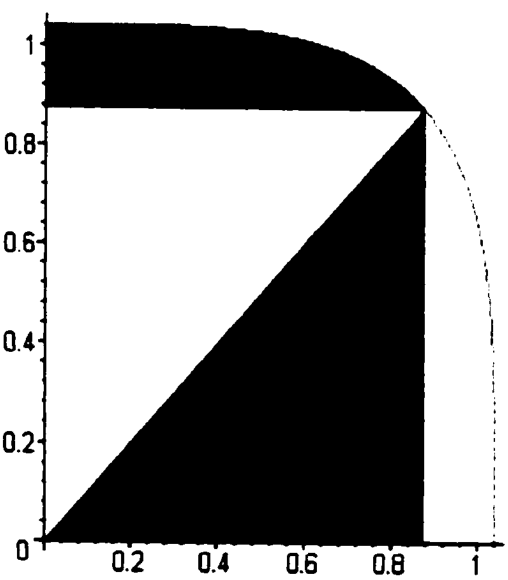

Figure 10.5: Fourth Order Cap Reduced Integration Region

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CHAPTER 11

MEASURE OF PLANES SEPARATING A TWO-PARAMETER CLASS OF COAXIAL PARABOLOIDS

11.1 Introduction

In this chapter we compute the measure of planes separating two paraboloids with graph of function parametrizations of the form

$$\mathbf{s_A}(x, y) = (x, y, -ax^2 - y^2) \quad \text{and}$$

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$$\mathbf{s_B}(x,y) = (-x, -y, x^2 + ay^2 + c)$$

where $(x, y, z) \in R^3$, $a > 1$, and $c > 0$. Thus this class of paraboloid pairs is parametrized by a shape parameter a and a separation parameter c.

Note that this example has discrete symmetry but not the continuous symmetry evident in the example of coaxial surfaces of revolution. It is an interesting example to study because it is a relatively simple example in which the set of separating double tangent planes have an empty intersection and hence the envelope of separating double tangent planes is not a cone.

11.2 Parametrization by Tangent Planes

Definition 11.2.1 *Let $(x, y, z) \in R^3$. Let S be a surface in R^3. A* **parametrization by tangent planes** *of S is a parametrization of the form*

$$\mathbf{s}(\theta, w) = (x(\theta, w), y(\theta, w), z(\theta, w))$$

where at a point (θ, w), θ is the angle with the x-axis of the projection onto the xy-plane of the upward normal vector and w is the z-coordinate of the intersection of the tangent plane with the z-axis.

Theorem 11.2.2 *Let $a > 1$. Let $c > 0$. Let $(x, y, z) \in R^3$. Parametrize two paraboloids as*

$$\mathbf{s_A}(x,y) = (x, y, -ax^2 - y^2) \tag{11.2.3}$$

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(11.2.4) $$\mathbf{s_B}(x,y) = (-x, -y, x^2 + ay^2 + c).$$

Then the paraboloids may be re-parametrized by tangent planes as

(11.2.5) $$\mathbf{s_A}(\theta, w) = \left(\frac{\sqrt{w}\cos\theta}{\sqrt{a(1+(a-1)\sin^2\theta)}}, \frac{\sqrt{aw}\sin\theta}{\sqrt{1+(a-1)\sin^2\theta}}, -w\right)$$

(11.2.6)
$$\mathbf{s_B}(\theta, w) = \left(-\frac{\sqrt{a(c-w)}\cos\theta}{\sqrt{1+(a-1)\cos^2\theta}}, -\frac{\sqrt{c-w}\sin\theta}{\sqrt{a(1+(a-1)\cos^2\theta)}}, 2c-w\right)$$

where at a point (θ, w), θ is the angle with the x-axis of the projection onto the xy-plane of the upward normal vector and w is the z-coordinate of the intersection of the tangent plane with the z-axis.

Proof. We differentiate Equation 11.2.5 to compute the tangent vectors to the coordinate curves of $\mathbf{s_A}$ at a point (x_A, y_A)

$$(1, 0, -2ax_A) \quad \text{and} \quad (0, 1, -2y_A).$$

Likewise the tangent vectors to the coordinate curves of $\mathbf{s_B}$ at a point (x_B, y_B) are

$$(-1, 0, 2x_B) \quad \text{and} \quad (0, -1, 2ay_B).$$

We then compute the crossproducts of the tangent vectors and normalize to get the corresponding upward unit normal vectors

$$\mathbf{n_A} = \frac{(2ax_A, 2y_A, 1)}{\sqrt{4a^2x_A^2 + 4y_A^2 + 1}} \quad \text{and}$$

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$$\mathbf{n_B} = \frac{(2x_B, 2ay_B, 1)}{\sqrt{4x_B^2 + 4a^2y_B^2 + 1}}.$$

Let ϕ_A and ϕ_B be the angles of the upward unit normal vectors to the respective surfaces with the positive z-axis for $0 \le \phi_A \le \pi/2$ and $0 \le \phi_B \le \pi/2$. Let θ_A and θ_B be the angles of the projections to the xy-plane of the respective normal vectors with the x-axis for $0 \le \theta_A < 2\pi$ and $0 \le \theta_B < 2\pi$. Then

$$\cos\phi_A = n_A \cdot (0, 0, 1) = \frac{1}{\sqrt{4a^2x_A^2 + 4y_A^2 + 1}} \quad \text{and}$$

$$\cos\theta_A = \frac{(ax_A, y_A, 0)}{\sqrt{a^2x_A^2 + y_A^2}} \cdot (1, 0, 0) = \frac{ax_A}{\sqrt{a^2x_A^2 + y_A^2}}.$$

Solving these two equations for x_A and y_A gives

$$x_A = \frac{\cos\theta_A \tan\phi_A}{2a} \quad \text{and} \tag{11.2.7}$$

$$y_A = \frac{\sin\theta_A \tan\phi_A}{2}. \tag{11.2.8}$$

Likewise

$$\cos\phi_B = n_B \cdot (0, 0, 1) = \frac{1}{\sqrt{4x_B^2 + 4a^2y_B^2 + 1}} \quad \text{and}$$

$$\cos\theta_B = \frac{(x_B, ay_B, 0)}{\sqrt{x_B^2 + a^2y_B^2}} \cdot (1, 0, 0) = \frac{x_B}{\sqrt{x_B^2 + a^2y_B^2}}.$$

Solving these two equations for x_B and y_B gives

$$x_B = \frac{\cos\theta_B \tan\phi_B}{2} \quad \text{and} \tag{11.2.9}$$

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$$(11.2.10) \qquad y_B = \frac{\sin\theta_B \tan\phi_B}{2a}.$$

Let w_A and w_B represent the intersections of the tangent planes to the respective surfaces at the respective points (x_A, y_A) and (x_B, y_B) with the z-axis. For fixed θ_A and w_A find the value of ϕ_A for tangent planes to the first surface as follows. We first compute

$$\mathbf{n_A} \cdot \mathbf{s_A} = \frac{ax_A^2 + y_A^2}{\sqrt{4a^2x_A^2 + 4y_A^2 + 1}} \quad \text{and}$$

$$\mathbf{n_A} \cdot (0, 0, w_A) = \frac{w_A}{\sqrt{4a^2x_A^2 + 4y_A^2 + 1}}.$$

Then solving $\mathbf{n_A} \cdot \mathbf{s_A} = \mathbf{n_A} \cdot (0, 0, w_A)$ for ϕ_A gives

$$(11.2.11) \qquad \tan\phi_A = \frac{2\sqrt{aw_A}}{\sqrt{1 + (a-1)\sin^2\theta_A}}.$$

Similarly for fixed θ_B and w_B we find the value of ϕ_B for tangent planes to the second surface as follows. We first compute

$$\mathbf{n_B} \cdot \mathbf{s_B} = \frac{-x_B^2 - ay_B^2}{\sqrt{4x_B^2 + 4a^2y_B^2 + 1}} \quad \text{and}$$

$$\mathbf{n_B} \cdot (0, 0, w_B) = \frac{w_B}{\sqrt{4x_B^2 + 4a^2y_B^2 + 1}}.$$

Then solving $\mathbf{n_B} \cdot \mathbf{s_B} = \mathbf{n_B} \cdot (0, 0, w_B)$ for ϕ_B gives

$$(11.2.12) \qquad \tan\phi_B = \frac{2\sqrt{a(c - w_B)}}{\sqrt{1 + (a-1)\cos^2\theta_B}}.$$

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Then substituting equation 11.2.11 into equation 11.2.7 and equation 11.2.12 into equation 11.2.9 yields

$$x_A = \frac{\sqrt{w_A}\cos\theta_A}{\sqrt{a(1+(a-1)\sin^2\theta_A)}}, \qquad y_A = \frac{\sqrt{aw_A}\sin\theta_A}{\sqrt{1+(a-1)\sin^2\theta_A}} \quad \text{and}$$

$$x_B = \frac{\sqrt{a(c-w_B)}\cos\theta_B}{\sqrt{1+(a-1)\cos^2\theta_B}}, \qquad y_B = \frac{\sqrt{c-w_B}\sin\theta_B}{\sqrt{a(1+(a-1)\sin^2\theta_B)}}.$$

Then substituting these into the equations 11.2.3 and 11.2.4 of the original parametrization yields the desired equations 11.2.5 and 11.2.6. □

11.3 The Envelope of Separating Double Tangent Planes

Lemma 11.3.1 *Let $a > 1$. Let $c > 0$. Let $(x, y, z) \in R^3$. Parametrize two paraboloids as*

$$\mathbf{s_A}(x, y) = (x, y, -ax^2 - y^2)$$

$$\mathbf{s_B}(x, y) = (-x, -y, x^2 + ay^2 + c).$$

Then the locus of separating double tangent points on surface $\mathbf{s_A}$ is given by

$$\theta \to \left(\frac{\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{\sqrt{ac}\sin\theta}{\sqrt{a+1}}, -\frac{c(1+(a-1)\sin^2\theta)}{a+1}\right)$$

and the locus of corresponding separating double tangent points on surface $\mathbf{s_B}$ is given by

$$\theta \to \left(-\frac{\sqrt{ac}\cos\theta}{\sqrt{a+1}}, -\frac{\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(a+2+(a-1)\cos^2\theta)}{a+1}\right).$$

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Proof. By Theorem 11.2.2 the surfaces may be reparametrized by an angle μ and intersection with the z-axis w of the tangent planes as

$$\mathbf{s_A}(\theta, w) = \left(\frac{\sqrt{w}\cos\theta}{\sqrt{a(1+(a-1)\sin^2\theta)}}, \frac{\sqrt{aw}\sin\theta}{\sqrt{1+(a-1)\sin^2\theta}}, -w\right)$$

$$\mathbf{s_B}(\theta, w) = \left(-\frac{\sqrt{a(c-w)}\cos\theta}{\sqrt{1+(a-1)\cos^2\theta}}, -\frac{\sqrt{c-w}\sin\theta}{\sqrt{a(1+(a-1)\sin^2\theta)}}, 2c-w\right).$$

Recall equations 11.2.11 and 11.2.12

$$\tan\phi_A = \frac{2\sqrt{aw_A}}{\sqrt{1+(a-1)\sin^2\theta_A}}$$

$$\tan\phi_B = \frac{2\sqrt{a(c-w_B)}}{\sqrt{1+(a-1)\cos^2\theta_B}}$$

giving the angles of the normal vectors with the z-axis at the points $\mathbf{s_A}(\theta_A, w_A)$ and $\mathbf{s_B}(\theta_B, w_B)$. Because the reparametrization of the surface was by tangent planes, in order for corresponding points on the two surfaces to be points of double tangency, the corresponding coordinates must be the same. Thus we can drop the subscripts to get

$$\tan\phi = \frac{2\sqrt{aw}}{\sqrt{1+(a-1)\sin^2\theta}}$$

$$\tan\phi = \frac{2\sqrt{a(c-w)}}{\sqrt{1+(a-1)\cos^2\theta}}.$$

Solving the system for w then gives

$$w = w(\theta) = \frac{c(1+(a-1)\sin^2\theta)}{a+1}. \tag{11.3.2}$$

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Thus the locus of separating double tangent points on surface $\mathbf{s_A}$ is given by

$$\theta \to \mathbf{s_A}(\theta, w(\theta)) = \left(\frac{\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{\sqrt{ac}\sin\theta}{\sqrt{a+1}}, -\frac{c(1+(a-1)\sin^2\theta)}{a+1}\right)$$

and the locus of corresponding separating double tangent points on surface $\mathbf{s_B}$ is given by

$$\theta \to \mathbf{s_B}(\theta, w(\theta)) = \left(-\frac{\sqrt{ac}\cos\theta}{\sqrt{a+1}}, -\frac{\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(a+2+(a-1)\cos^2\theta)}{a+1}\right)$$

which proves the lemma. □

Theorem 11.3.3 *Let $a > 1$. Let $c > 0$. Let $0 \le \theta < 2\pi$. Let $0 \le t \le 1$. Let $(x, y, z) \in R^3$. Parametrize two paraboloids as*

$$\mathbf{s_A}(x, y) = (x, y, -ax^2 - y^2)$$

$$\mathbf{s_B}(x, y) = (-x, -y, x^2 + ay^2 + c).$$

Then the envelope of separating double tangent planes may be parametrized as $\mathbf{s}(\theta, t) =$

$$\left(\frac{(1-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{(a-(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(2(a+1)t-1+(1-a)\sin^2\theta)}{a+1}\right).$$

Proof. According to Lemma 11.3.1 the corresponding points of separating double tangency on the two surfaces are given by

$$\mathbf{s_A}(\theta, w(\theta)) = \left(\frac{\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{\sqrt{ac}\sin\theta}{\sqrt{a+1}}, -\frac{c(1+(a-1)\sin^2\theta)}{a+1}\right) \quad \text{and}$$

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$$\mathbf{s_B}(\theta, w(\theta)) = \left(-\frac{\sqrt{ac}\cos\theta}{\sqrt{a+1}}, -\frac{\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(a+2+(a-1)\cos^2\theta)}{a+1}\right).$$

Let

$$\mathbf{s}(\theta, t) = (1-t)\mathbf{s_A}(\theta, w(\theta)) + t\mathbf{s_B}(\theta, w(\theta)) =$$

$$\left(\frac{(1-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{(a-(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(2(a+1)t-1+(1-a)\sin^2\theta)}{a+1}\right).$$

This is our candidate for the envelope. Recall that an envelope of separating double tangent planes is defined as a surface whose tangent planes are exactly the set of separating double tangent planes. We check that our candidate is an envelope as follows. First recall Equation 11.2.5

$$\mathbf{s_A}(\theta, w) = (\frac{\sqrt{w}\cos\theta}{\sqrt{a(1+(a-1)\sin^2\theta)}}, \frac{\sqrt{aw}\sin\theta}{\sqrt{1+(a-1)\sin^2\theta}}, -w)$$

giving the equation of the lower surface parametrized by tangent planes. We differentiate to get the equations

$$\mathbf{s_{A\theta}}(\theta, w) = \frac{\sqrt{aw}}{(1+(a-1)sin^2\theta)^{3/2}}(-\sin\theta, \cos\theta, 0) \quad \text{and}$$

$$\mathbf{s_{Aw}}(\theta, w) = \left(\frac{\cos\theta}{2\sqrt{aw(1+(a-1)sin^2\theta)}}, \frac{\sqrt{a}\sin\theta}{2\sqrt{w(1+(a-1)\sin^2\theta}}, -1\right)$$

of the tangent vectors parallel to the coordinate curves. Then taking the cross product and normalizing gives the upward unit normal vector

$$\mathbf{n_A}(\theta, w) = \frac{(2\sqrt{az}\cos\theta, 2\sqrt{az}\sin\theta, \sqrt{1+(a-1)\sin^2\theta})}{\sqrt{4az+1+(a-1)\sin^2\theta}}.$$

Then using Equation 11.3.2 to replace w by

$$w(\theta) = \frac{c(1+(a-1)\sin^2\theta)}{a+1}$$

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we get

$$\mathbf{n_A}(\mu, w(\theta)) = \frac{(2\sqrt{ac}\ \cos\theta, 2\sqrt{ac}\ \sin\theta, \sqrt{a+1})}{\sqrt{4ac+a+1}}$$

at points of double tangency. Thus the distance of the separating double tangent planes from the origin is given by

$$\mathbf{n_A}(\theta, w(\theta)) \cdot \mathbf{s_A}(\theta, w(\theta))$$

$$= \frac{(2\sqrt{ac}\ \cos\theta, 2\sqrt{ac}\ \sin\theta, \sqrt{a+1})}{\sqrt{4ac+a+1}} \cdot \left(\frac{\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{\sqrt{ac}\sin\theta}{\sqrt{a+1}}, -\frac{c(1+(a-1)\sin^2\theta)}{a+1}\right)$$

$$= \frac{c(1+(a-1)\sin^2\theta)}{\sqrt{a+1}\sqrt{4ac+a+a}}.$$

Similarly we differentiate to get the tangent vectors to the coordinate curves of our candidate for the envelope to get

$$\mathbf{s}_\theta = \left(-\frac{(1-(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{(a-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{2c(1-a)\sin\theta\cos\theta}{a+1}\right)$$

$$\text{and}\quad \mathbf{s}_t = \left(\frac{-\sqrt{a+1}\sqrt{c}\cos\theta}{\sqrt{a}}, \frac{-\sqrt{a+1}\sqrt{c}\sin\theta}{\sqrt{a}}, 2c\right).$$

Then we take the cross product and normalize to get the normal vector

$$\mathbf{n}(\theta, t) = \frac{(2\sqrt{ac}\ \cos\theta, 2\sqrt{ac}\ \sin\theta, \sqrt{a+1})}{\sqrt{4ac+a+1}}.$$

Thus the distance of planes tangent to our envelope candidate from the origin is

$$\mathbf{n}(\theta, t) \cdot \mathbf{s}(\theta, t)$$

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$$= \frac{(2\sqrt{ac}\ \cos\theta, 2\sqrt{ac}\ \sin\theta, \sqrt{a+1})}{\sqrt{4ac+a+1}}.$$

$$\left(\frac{(1-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{(a-(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(2(a+1)t-1+(1-a)\sin^2\theta)}{a+1}\right)$$

$$= \frac{c(1+(a-1)\sin^2\theta)}{\sqrt{a+1}\sqrt{4ac+a+a}}.$$

Thus for $0 \leq \theta < 2\pi$ the normal directions and distances to the origin of the tangent planes to the envelope candidate agree with the normal directions and distances to the origin of the separating double tangent planes. Thus our candidate for the envelope is in fact an envelope of the set of separating double tangent planes. □

See Section 11.6 for a graph of the envelope when $a = 2$ and $c = 6$.

11.4 Direct Measure

Theorem 11.4.1 *Let $a > 1$. Let $c > 0$. Let $\mathbf{s_A}(x, y) = (x, y, -ax^2 - y^2)$. Let $\mathbf{s_B}(x, y) = (-x, -y, x^2 + ay^2 + c)$. Then the measure of planes separating the two surfaces is*

$$\frac{\pi}{4a}\left[4ac + (a+1)\ln\left(\frac{a+1}{4ac+a+1}\right)\right].$$

Proof. In $\theta w\phi$-coordinates the motion-invariant measure on the set of planes in R^3 is given by Ambartzumian (1990, 53) as $\cos\phi\ \sin\phi\ d\phi\ dw\ d\theta$. Recall from equation 11.3.2 that at separating double tangent points

$$w(\theta) = \frac{c(1+(a-1)\sin^2\theta)}{a+1}.$$

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Recall also equations 11.2.11 and 11.2.12

$$\tan\phi_A = \frac{2\sqrt{aw_A}}{\sqrt{1+(a-1)\sin^2\theta_A}}$$

$$\tan\phi_B = \frac{2\sqrt{a(c-w_B)}}{\sqrt{1+(a-1)\cos^2\theta_B}}$$

giving the angles of the normal vectors with the z-axis at the points of tangency. Thus the measure of planes separating the two surfaces is

$$\int_0^{2\pi}\int_0^{\frac{c(1+(a-1)\sin^2\theta)}{a+1}}\int_0^{\tan^{-1}\frac{2\sqrt{aw}}{\sqrt{1+(a-1)\sin^2\theta}}} \cos\phi\sin\phi d\phi\ dw\ d\theta$$

$$+\int_0^{2\pi}\int_{\frac{c(1+(a-1)\sin^2\theta)}{a+1}}^{c}\int_0^{\tan^{-1}\frac{2\sqrt{a(c-w)}}{\sqrt{1+(a-1)\cos^2\theta}}} \cos\phi\sin\phi d\phi\ dw\ d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{c(1+(a-1)\sin^2\theta)}{a+1}} \frac{2aw}{4aw+1+(a-1)\sin^2\theta}dw\ d\theta$$

$$+\int_0^{2\pi}\int_{\frac{c(1+(a-1)\sin^2\theta)}{a+1}}^{c} \frac{2a(c-w)}{4a(c-w)+1+(a-1)\cos^2\theta}dw\ d\theta$$

$$=\frac{\pi}{4a}\left[4ac+(a+1)\ln\left(\frac{a+1}{4ac+a+1}\right)\right]$$

which is what we were trying to prove. □

11.5 Mean Curvature Integral

Lemma 11.5.1 *The total absolute mean curvature over the envelope is $2\pi c$.*

Proof. As above the envelope of separating double tangent planes may be parametrized as

$$\mathbf{s}(\theta, t) =$$

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$$\left(\frac{(1-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{(a-(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{c(2(a+1)t-1+(1-a)\sin^2\theta)}{a+1}\right).$$

We differentiate to compute the tangent vectors of the coordinate curves

$$\mathbf{s}_\theta(\theta,t) = \left(\frac{-(1-(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{(a-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{2c(1-a)\sin\theta\cos\theta}{a+1}\right)$$

$$\text{and} \quad \mathbf{s}_t(\theta,t) = \left(\frac{-\sqrt{c(a+1)}\cos\theta}{\sqrt{a}}, \frac{-\sqrt{c(a+1)}\sin\theta}{\sqrt{a}}, 2c\right).$$

Then we take the cross product of the tangent vectors to find the normal direction

$$\left(\frac{2c^{3/2}\cos\theta(a-(a+1)t+(1-a)\sin^2\theta}{\sqrt{a(a+1)}}, \frac{2c^{3/2}\sin\theta(1-(a+1)t+(a-1)\cos^2\theta}{\sqrt{a(a+1)}},\right.$$
$$\left.\frac{c}{a}(1+(a-1)\cos^2\theta-(a+1)t)\right).$$

We then compute the norm of the normal direction

$$\frac{c(1-(a+1)t+(a-1)\cos^2\theta)\sqrt{4ac+a+1}}{a\sqrt{a+1}}$$

and then divide the normal direction by its norm to get the unit normal vector

$$\mathbf{n}(\theta,t) = \frac{(2\sqrt{ac}\ \cos\theta, 2\sqrt{ac}\ \sin\theta, \sqrt{a+1})}{\sqrt{4ac+a+1}}.$$

We next compute the coefficients E, F, and G of the first fundamental form

$$\mathrm{E} = \mathbf{s}_\theta \cdot \mathbf{s}_\theta$$

$$= \frac{c}{a(a+1)^2}\Big[(a+1)\Big((a+1)t^2+\sin^2\theta+a^2\cos^2\theta-2(a+1)t(\sin^2\theta+a\cos^2\theta)\Big)$$
$$+4ac(a-1)^2(\cos^2\theta-\cos^4\theta)\Big],$$

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$$\mathrm{F} = \mathbf{s}_\theta \cdot \mathbf{s}_t = c(1-a)\left(\frac{4ac+a+1}{a(a+1)}\right)\sin\theta\cos\theta,$$

$$\text{and}\quad \mathrm{G} = \mathbf{s}_t \cdot \mathbf{s}_t = \frac{c}{a}(4ac+a+1).$$

We next compute the derivatives of the tangent vectors

$$\mathbf{s}_{\theta\theta} = \left(\frac{-(1-(a+1)t)\sqrt{c}\cos\theta}{\sqrt{a(a+1)}}, \frac{-(a_(a+1)t)\sqrt{c}\sin\theta}{\sqrt{a(a+1)}}, \frac{2c(1-a)(\cos^2\theta - \sin^2\theta)}{a+1}\right),$$

$$\mathbf{s}_{\theta t} = \frac{\sqrt{c(a+1)}}{\sqrt{a}}(\sin\theta, -\cos\theta, 0),$$

$$\text{and}\quad \mathbf{s}_{tt} = (0, 0, 0).$$

Thus the coefficients e, f, and g of the second fundamental form are

$$e = \mathbf{n}\cdot\mathbf{s}_{\theta\theta} = \frac{-2c}{\sqrt{a+1}\sqrt{4ac+a+1}}\left(1 + (a-1)\cos^2\theta - (a+1)t\right),$$

$$f = \mathbf{n}\cdot\mathbf{s}_{\theta t} = 0,$$

$$\text{and}\quad g = \mathbf{n}\cdot\mathbf{s}_{tt} = 0.$$

Thus the integrand of the absolute mean curvature over the envelope is

$$|\mathrm{H}|dS = |\mathrm{H}||\mathbf{n}|dt\ d\theta = \frac{|e\mathrm{G} - 2f\mathrm{F} + g\mathrm{E}|}{2|\mathbf{n}|}dt\ d\theta = c\ dt\ d\theta.$$

Thus the total absolute mean curvature over the envelope is

$$\int_0^{2\pi}\int_0^1 c\ dt\ d\theta = 2\pi c.$$

□

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Lemma 11.5.2 *The total absolute mean curvature over the lower cap is*

$$\frac{\pi}{8a}\left[4ac+(a+1)\ln\left(\frac{4ac+a+1}{a+1}\right)\right].$$

Proof of lemma. We parametrize the lower cap by tangent planes

$$\mathbf{s_A}(\theta, w) = \left(\frac{\sqrt{w}\cos\theta}{\sqrt{a(1+(a-1)\sin^2\theta)}}, \frac{\sqrt{aw}\sin\theta}{\sqrt{1+(a-1)\sin^2\theta}}, -w\right).$$

We then differentiate to find the tangent vectors to the coordinate curves

$$\mathbf{s_{A\theta}}(\theta, w) = \frac{\sqrt{aw}}{(1+(a-1)\sin^2\theta)^{3/2}}(-\sin\theta, \cos\theta, 0) \quad \text{and}$$

$$\mathbf{s_{Aw}}(\theta, w) = \left(\frac{\cos\theta}{2\sqrt{aw(1+(a-1)\sin^2\theta)}}, \frac{\sqrt{a}\sin\theta}{2\sqrt{w(1+(a-1)\sin^2\theta)}}, -1\right).$$

Next we compute the normal direction by taking the cross product

$$\left(\frac{-\sqrt{aw}\cos\theta}{(1+(a-1)\sin^2\theta)^{(3/2)}}, \frac{-\sqrt{aw}\sin\theta}{(1+(a-1)\sin^2\theta)^{(3/2)}}, \frac{-1}{2(1+(a-1)\sin^2\theta)^{(3/2)}}\right).$$

We then compute the norm

$$\frac{\sqrt{4aw+1+(a-1)\sin^2\theta}}{2(1+(a-1)\sin^2\theta)^{3/2}}.$$

We then compute the unit upward normal vector by dividing by the negative of the norm to get

$$\mathbf{n_A} = \frac{(2\sqrt{aw}\cos\theta, 2\sqrt{aw}\sin\theta, \sqrt{1+(a-1)\sin^2\theta})}{\sqrt{4aw+1+(a-1)\sin^2\theta}}.$$

Next we compute the coefficients E, F, and G of the first fundamental form

$$\mathrm{E} = \mathbf{s_{A\theta}}\cdot\mathbf{s_{A\theta}} = \frac{aw}{(1+(a-1)\sin^2\theta)^3}.$$

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$$\mathrm{F} = \mathbf{s}_{\mathbf{A}\theta} \cdot \mathbf{s}_{\mathbf{Aw}} = \frac{(a-1)\sin\theta\cos\theta}{2(1+(a-1)\sin^2\theta)^2}, \quad \text{and}$$

$$\mathrm{G} = \mathbf{s}_{\mathbf{Aw}} \cdot \mathbf{s}_{\mathbf{Aw}} = \frac{1+(a^2-1)\sin^2\theta + 4aw(1+(a-1)\sin^2\theta)}{4aw(1+(a-1)\sin^2\theta)}.$$

Next we differentiate the tangent vectors to get

$$\mathbf{s}_{\mathbf{A}\theta\theta} = \frac{\sqrt{aw}}{(1+(a-1)\sin^2\theta)^{5/2}}\Big(\cos\theta(-1+2(a-1)\sin^2\theta), \sin\theta(a+2(a-1)\cos^2\theta), 0\Big).$$

$$\mathbf{s}_{\mathbf{A}\theta\mathbf{w}} = \frac{\sqrt{a}}{2\sqrt{w}(1+(a-1)\sin^2\theta)^{3/2}}\big(-\sin\theta, \cos\theta, 0\big), \quad \text{and}$$

$$\mathbf{s}_{\mathbf{Aww}} = \frac{-1}{4\sqrt{a}w^{3/2}\sqrt{1+(a-1)\sin^2\theta}}\big(\cos\theta, a\sin\theta, 0\big).$$

We then compute the coefficients e, f, and g of the second fundamental form

$$e = \mathbf{n}_{\mathbf{A}} \cdot \mathbf{s}_{\mathbf{A}\theta\theta} = \frac{2aw}{(1+(a-1)\sin^2\theta)^{3/2}\sqrt{4aw+1+(a-1)\sin^2\theta}}.$$

$$f = \mathbf{n}_{\mathbf{A}} \cdot \mathbf{s}_{\mathbf{A}\theta\mathbf{w}} = 0, \quad \text{and}$$

$$g = \mathbf{n}_{\mathbf{A}} \cdot \mathbf{s}_{\mathbf{Aww}} = \frac{\sqrt{1+(a-1)\sin^2\theta}}{2w\sqrt{4aw+a+(a-1)\sin^2\theta}}.$$

Thus the integrand of the absolute mean curvature over the lower cap is

$$|\mathrm{H}|dS = |\mathrm{H}||\mathbf{n}_{\mathbf{A}}|dw\,d\theta = \frac{|e\mathrm{G} - 2f\mathrm{F} + g\mathrm{E}|}{2|\mathbf{n}_{\mathbf{A}}|}dw\,d\theta$$

$$= \frac{4aw+a+1}{2(4aw+1+(a-1)\sin^2\theta)}\,dw\,d\theta.$$

Thus the total absolute mean curvature over the lower cap is

$$\int_0^{2\pi}\int_0^{\frac{2}{a+1}(1+(a-1)\sin^2\theta)} \frac{4aw+a+1}{2(4aw+1+(a-1)\sin^2\theta)}\,dw\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{2}{a+1}(1+(a-1)\sin^2\theta)} 1 + \frac{a+(1-a)\sin^2\theta}{4aw+1+(a-1)\sin^2\theta}\,dw\,d\theta$$

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$$= \frac{\pi}{8a}\left[4ac + (a+1)\ln\left(\frac{4ac+a+1}{a+1}\right)\right].$$

□

Lemma 11.5.3 *The total absolute mean curvature over the upper cap is*

$$\frac{\pi}{8a}\left[4ac + (a+1)\ln\left(\frac{4ac+a+1}{a+1}\right)\right].$$

Proof of lemma. We parametrize the upper cap by tangent planes

$$\mathbf{s_B}(\theta, w) = \left(\frac{-\sqrt{a(c-w)}\cos\theta}{\sqrt{1+(a-1)\sin^2\theta}}, -\frac{\sqrt{c-w}\sin\theta}{\sqrt{a(1+(a-1)\sin^2\theta)}}, 2c-w\right).$$

We then differentiate to find the tangent vectors to the coordinate curves

$$\mathbf{s_{B\theta}}(\theta, w) = \frac{\sqrt{a(c-w)}}{(1+(a-1)\cos^2\theta)^{3/2}}(\sin\theta, -\cos\theta, 0)$$

and $\mathbf{s_{Bw}}(\mu, w)$

$$= \left(\frac{\sqrt{a}\cos\theta}{2\sqrt{(c-w)(1+(a-1)\cos^2\theta)}}, \frac{\sin\theta}{2\sqrt{a(c-w)(1+(a-1)\cos^2\theta)}}, -1\right).$$

Next we compute the normal direction by taking the cross product

$$\frac{(2\sqrt{a(c-w)}\cos\theta, 2\sqrt{a(c-w)}\sin\theta, \sqrt{1+(a-1)\cos^2\theta})}{2(1+(a-1)\cos^2\theta)^{3/2}}.$$

We then compute the norm

$$\frac{\sqrt{4a(c-w)+1+(a-1)\cos^2\theta}}{2(1+(a-1)\cos^2\theta)^{3/2}}.$$

We then compute the unit upward normal vector by dividing by the norm to get

$$\mathbf{n_B} = \frac{(2\sqrt{a(c-w)}\cos\theta, 2\sqrt{a(c-w)}\sin\theta, \sqrt{1+(a-1)\cos^2\theta})}{\sqrt{4a(c-w)+1+(a-1)\cos^2\theta}}.$$

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Next we compute the coefficients E, F, and G of the first fundamental form

$$\mathrm{E} = \mathbf{s_{B\theta}} \cdot \mathbf{s_{B\theta}} = \frac{a(c-w)}{(1+(a-1)\cos^2\theta)^3},$$

$$\mathrm{F} = \mathbf{s_{B\theta}} \cdot \mathbf{s_{Bw}} = \frac{(a-1)\sin\theta\cos\theta}{2(1+(a-1)\cos^2\theta)^2}, \quad \text{and}$$

$$\mathrm{G} = \mathbf{s_{Bw}} \cdot \mathbf{s_{Bw}} = \frac{1+(a^2-1)\cos^2\theta + 4a(c-w)(1+(a-1)\cos^2\theta)}{4a(c-w)(1+(a-1)\cos^2\theta)}.$$

Next we differentiate the tangent vectors to get

$$\mathbf{s_{B\theta\theta}} = \frac{\sqrt{a(c-w)}}{(1+(a-1)\cos^2\theta)^{5/2}}\Big(\cos\theta(a+2(a-1)\sin^2\theta), \sin\theta(1-2(a-1)\cos^2\theta), 0\Big),$$

$$\mathbf{s_{B\theta w}} = \frac{\sqrt{a}}{2\sqrt{c-w}(1+(a-1)\cos^2\theta)^{3/2}}\big(-\sin\theta, \cos\theta, 0\big), \quad \text{and}$$

$$\mathbf{s_{Bww}} = \frac{1}{4\sqrt{a}(c-w)^{3/2}\sqrt{1+(a-1)\cos^2\theta}}\big(a\cos\theta, \sin\theta, 0\big).$$

We then compute the coefficients e, f, and g of the second fundamental form

$$e = \mathbf{n_B} \cdot \mathbf{s_{B\theta\theta}} = \frac{2a(c-w)}{(1+(a-1)\cos^2\theta)^{3/2}\sqrt{4a(c-w)+1+(a-1)\cos^2\theta}},$$

$$f = \mathbf{n_B} \cdot \mathbf{s_{B\theta w}} = 0, \quad \text{and}$$

$$g = \mathbf{n_B} \cdot \mathbf{s_{Bww}} = \frac{\sqrt{1+(a-1)\cos^2\theta}}{2(c-w)\sqrt{4a(c-w)+a+(a-1)\cos^2\theta}}.$$

Thus the integrand of the absolute mean curvature over the upper cap is

$$|\mathrm{H}|dS = |\mathrm{H}||\mathbf{n_B}|dw\, d\theta = \frac{|e\mathrm{G} - 2f\mathrm{F} + g\mathrm{E}|}{2|\mathbf{n_B}|}dw\, d\theta$$

$$= \frac{4a(c-w)+a+1}{2(4a(c-w)+1+(a-1)\cos^2\theta)}\, dw\, d\theta.$$

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Thus the total absolute mean curvature over the upper cap is

$$\int_0^{2\pi}\int_{\frac{1}{a+1}(1+(a-1)\sin^2\theta)}^{c} \frac{4a(c-w)+a+1}{2(4a(c-w)+1+(a-1)\cos^2\theta)}\,dw\,d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\int_{\frac{1}{a+1}(1+(a-1)\sin^2\theta)}^{c} 1+\frac{a+(1-a)\cos^2\theta}{4a(c-w)+1+(a-1)\cos^2\theta}\,dw\,d\theta$$

$$=\frac{\pi}{8a}\left[4ac+(a+1)\ln\left(\frac{4ac+a+1}{a+1}\right)\right].$$

□

Theorem 11.5.4 *The measure of planes separating the two paraboloids is equal to the total absolute mean curvature over the envelope minus the total mean curvature over the caps.*

Proof. Applying the lemmas above

$$\int_{S_E}|H|dS-\int_{C_A}|H|dS-\int_{C_B}|H|dS$$

$$=2\pi c-\frac{\pi}{8a}\left[4ac+(a+1)\ln\left(\frac{4ac+a+1}{a+1}\right)\right]-\frac{\pi}{8a}\left[4ac+(a+1)\ln\left(\frac{4ac+a+1}{a+1}\right)\right]$$

$$=\frac{\pi}{4a}\left[4ac+(a+1)\ln\left(\frac{a+1}{4ac+a+1}\right)\right]$$

which agrees with the measure of planes separating the two surfaces given in Theorem 11.4.1 above. Thus the measure of planes separating the two surfaces is equal to the total absolute mean curvature over the envelope minus the total mean curvature over the caps. □

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11.6 Figures

The Maple V Release 6 computer program was used to sketch the graph of the envelope when $a = 2$ and $c = 6$. In this case according to Theorem 11.3.3 the envelope may be parametrized as

$$\mathbf{s}(\theta, t) = \big((1-3t)\cos\theta, (2-3t)\sin\theta, 2(6t-1-\sin^2\theta)\big) \tag{11.6.1}$$

where $0 \le \theta < 2\pi$, $0 \le t \le 1$, and at a point (θ, t), θ is the angle of the projection onto the xy-plane of the upward normal vector with the x-axis for $(x, y, z) \in R^3$ and t represents a relative distance from the lower point of double tangency along the line segment connecting two corresponding points of double tangency. Note that this equation is linear in t and thus the envelope is a ruled surface. See Figure 11.1 for a graph of the envelope.

Figure 11.2 shows some important features of the envelope, namely the two separating double tangent curves

$$\mathbf{s}(\theta, 0) = \big(\cos\theta, 2\sin\theta, -2(1+\sin^2\theta)\big) \quad \text{and}$$

$$\mathbf{s}(\theta, 1) = \big(-2\cos\theta, -\sin\theta, 2(4+\cos^2\theta)\big)$$

and also the line of striction. According to Do Carmo (1976, 188-197) the line of striction is the unique directrix or generating curve of a non-cylindrical ruled surface that is perpendicular to the rulings of the surface. It is a property of the line of striction that any singularities of the surface are located on the line

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of striction. Following the procedure in Do Carmo the line of striction $\gamma(\theta)$ is computed as follows.

$$\gamma(\mu) = \mathbf{s}(\theta,0) - \frac{\mathbf{s}'(\theta,0)\cdot(\mathbf{s}'(\theta,1)-\mathbf{s}'(\theta,0))}{(\mathbf{s}'(\theta,1)-\mathbf{s}'(\theta,0))\cdot(\mathbf{s}'(\theta,1)-\mathbf{s}'(\theta,0))}(\mathbf{s}(\theta,1)-\mathbf{s}(\theta,0))$$

$$= \left(-\cos^3\theta, \sin^3\theta, 6\cos^2\theta\right).$$

As a start in understanding the singularities of the envelope we differentiate to find the tangent vector

$$3\sin\theta\cos\theta\left(\cos\theta, \sin\theta, -4\right)$$

of the line of striction. From this we see that the tangent vector is zero when

$$\mu = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

Furthermore, ignoring the common factor, we see that the tangent direction

$$\left(\cos\theta, \sin\theta, -4\right)$$

changes at these four points. Thus the line of striction has cusps at these four points.

In order to further understand the singularities and self-intersections of the envelope, the intersection of the envelope with the line of striction in the above parametrizations was computed as follows.

$$\mathbf{S}(\theta,t) = \gamma(\theta) \quad \text{implies}$$

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$$\begin{cases} (1-3t)\cos\theta &= -\cos^3\theta \\ (2-3t)\sin\theta &= \sin^3\theta \\ 2(6t-1-\sin^2\theta) &= 6\cos^2\theta \end{cases}$$

which implies $t = \frac{1}{3}(1+\cos^2\theta)$.

The Maple computer program was then used the to make separate graphs of the lower part of the envelope (for t between 0 and $(1+\cos^2\theta)/3$) and the upper part of the envelope (for t between $(1+\cos^2\theta)/3$ and 1) shown in Figures 11.3 and 11.4.

In order to determine which points on the line of striction are singularities, the normal vector near the line of striction is analyzed in greater detail. Equation 11.6.1 is differentiated to find the tangent vectors

$$\mathbf{s}_\theta(\theta, t) = \big((3t-1)\cos\theta, (2-3t)\sin\theta, -4\sin\theta\cos\theta\big) \quad \text{and}$$

$$\mathbf{s_t}(\theta, t) = (-3\cos\theta, -3\sin\theta, 12)$$

to the coordinate curves. The cross product is then taken to find the normal direction

$$(\theta, t) \rightarrow 3(1+\cos^2\theta - 3t)(4\cos\theta, 4\sin\theta, 1).$$

Thus on the upper envelope ($t > (1+\cos^2\theta)/3$) this normal direction is upward, on the lower envelope ($t < (1+\cos^2\theta)/3$) this normal direction is parallel but downward, and on the line of striction ($t = (1+\cos^2\mu)/3$) this normal direction

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is zero. Thus every point on the line of striction is a singularity and the points that are not cusps of the line of striction are folds of the envelope.

From these graphs it appears that the lower envelope has a self-intersection for arbitrary θ when the x-coordinate is zero. We then set the x-coordinate equal to zero and solve for t to get

$$(1-3t)\cos\theta = 0 \quad \text{which implies} \quad t = \frac{1}{3}.$$

Then replacing t with $1/3$ in the equation for the envelope produces the equation of the lower curve of self-intersection

$$\mu \rightarrow \left(0, \sin\theta, 2\cos^2\theta\right).$$

We note that θ and $\pi - \theta \bmod 2\pi$ have the same image on the lower curve of self-intersection and that θ and $\pi - \theta \bmod 2\pi$ are distinct for $\theta \neq \pi/2, 3\pi/2$. Note also that for $\theta = \pi/2, 3\pi/2$ the above equation produces a cusp of the line of striction. Thus the lower envelope self-intersects in an open curve whose endpoints are cusps of the line of striction.

It may seem odd not to classify these curves of self-intersection as singularities of the envelope but for $\theta \neq \pi/2, 3\pi/2$ the pairs (θ, t) and $(\pi - \theta, t)$ of points whose images are the same have non-overlapping neighborhoods in the domain of $\mathbf{s}$. Thus locally the function $\mathbf{s}$ is not singular at these points.

Likewise it appears that the upper envelope has a self-intersection for arbitrary μ when the y-coordinate is zero. We then set the y-coordinate equal

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to zero and solve for t to get

$$(2-3t)\sin\theta = 0 \quad \text{which implies} \quad t = \frac{2}{3}.$$

Then replacing t with $2/3$ in the equation for the envelope produces the equation of the upper curve of self-intersection

$$\theta \to \left(-\cos\theta, 0, 2(2+\cos^2\theta)\right).$$

We note that θ and $-\theta$ mod 2π have the same image on the upper curve of self-intersection and that θ and $-\theta$ mod 2π are distinct for $\mu \neq 0, \pi$. We also note that for $\theta = 0, \pi$ the above equation produces a cusp on the line of striction. Thus the upper envelope self-intersects in an open curve whose endpoints are cusps on the line of striction.

In summary the line of striction of the envelope is a closed piecewise smooth curve with four cusps. Each of the four smooth pieces of the line of striction is a fold of the envelope. Thus every point of the line of striction is a singularity of the envelope. Additionally the envelope has two open curves of self-intersection which are not points of singularity and whose endpoints are at the cusps of the line of striction.

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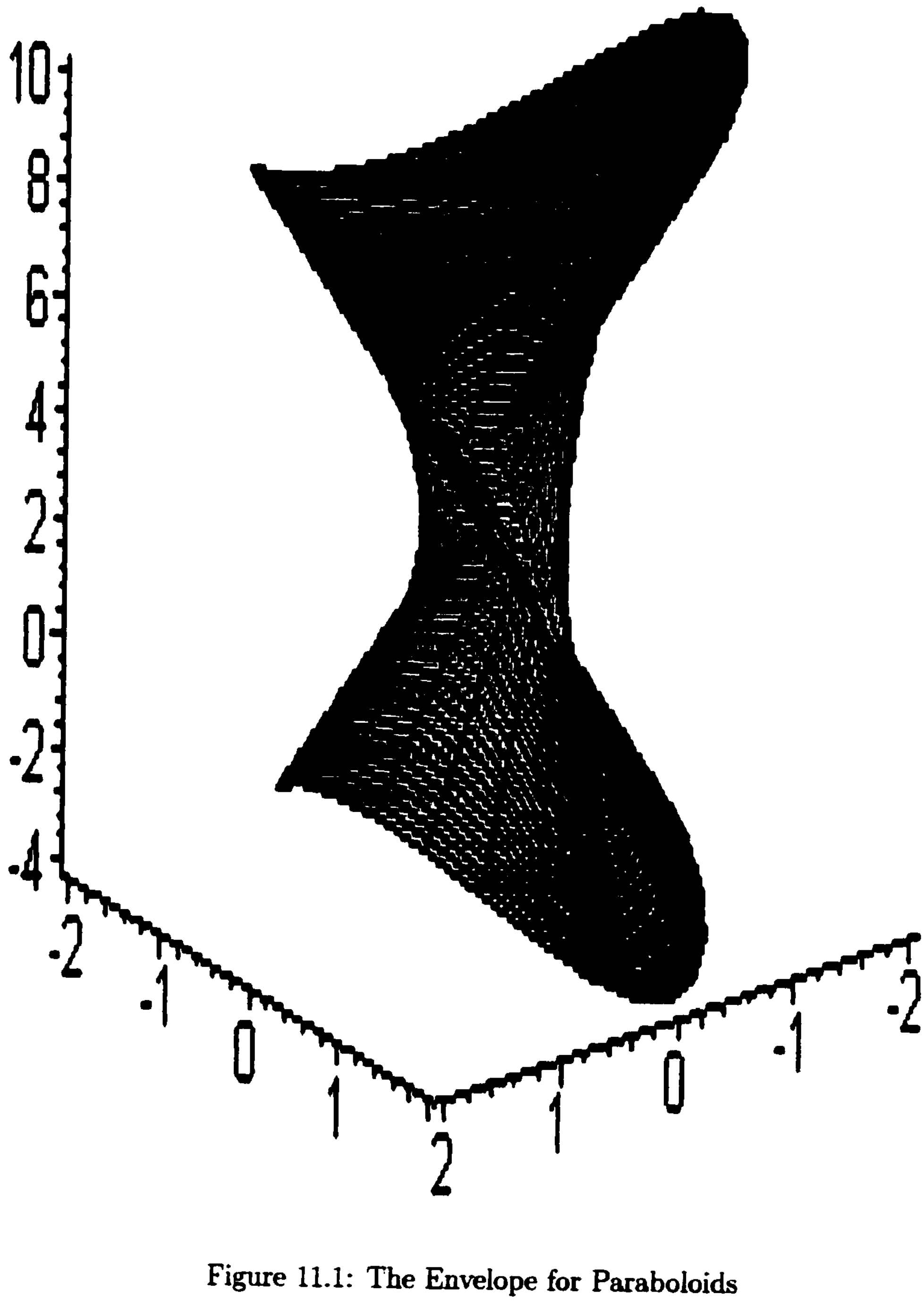

Figure 11.1: The Envelope for Paraboloids

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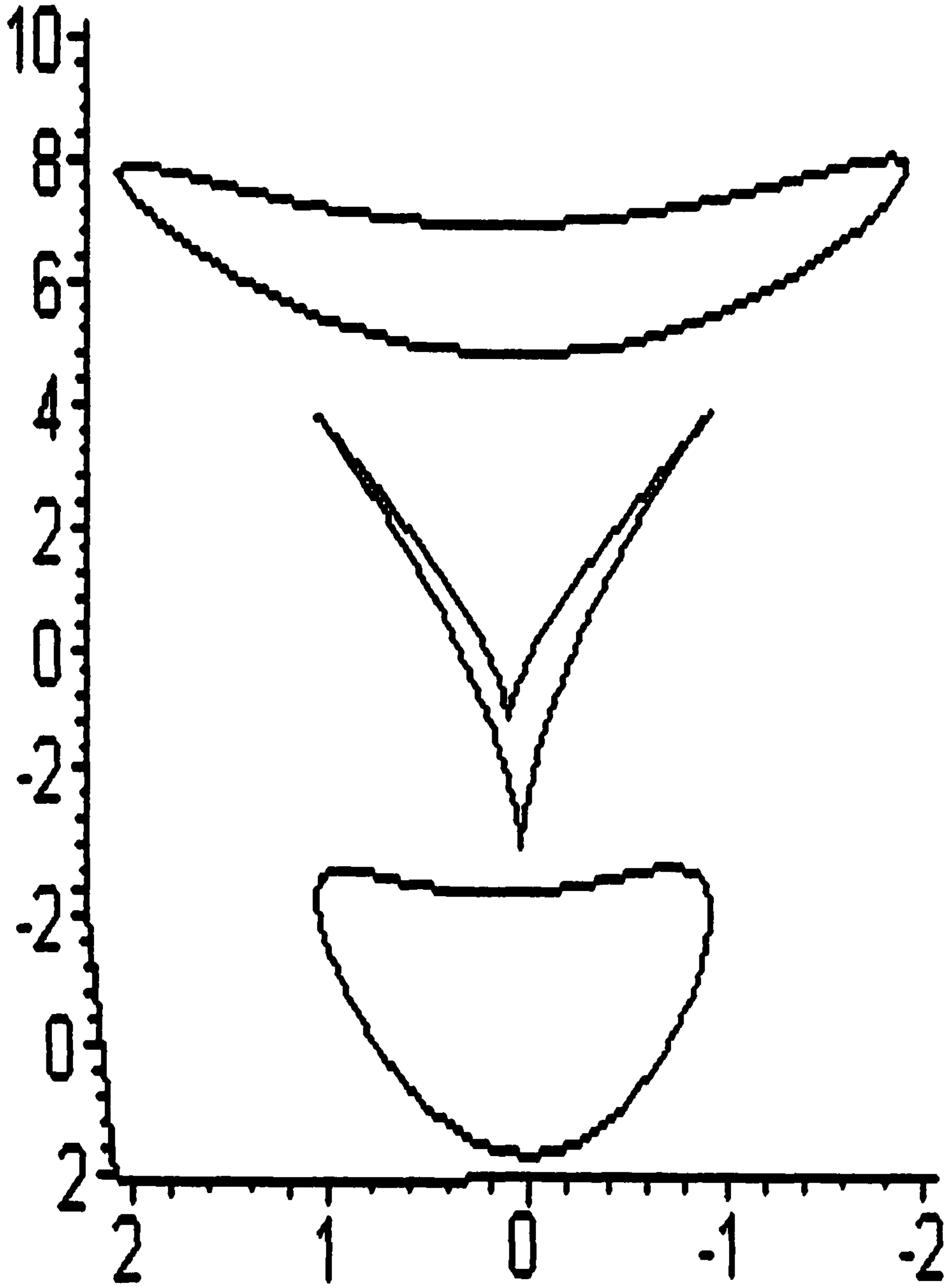

Figure 11.2: Envelope Boundaries for Paraboloids

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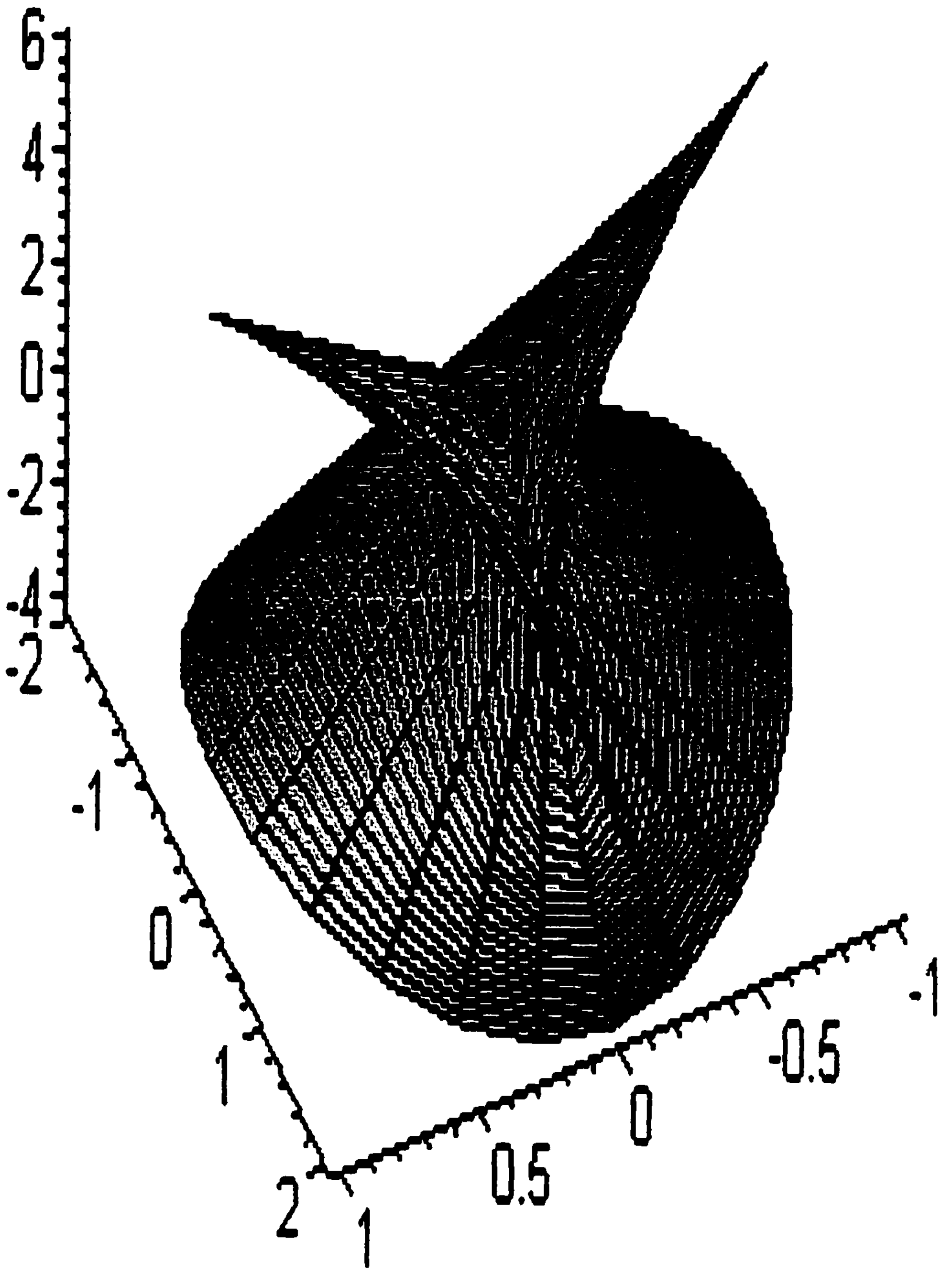

Figure 11.3: Lower Envelope for Paraboloids

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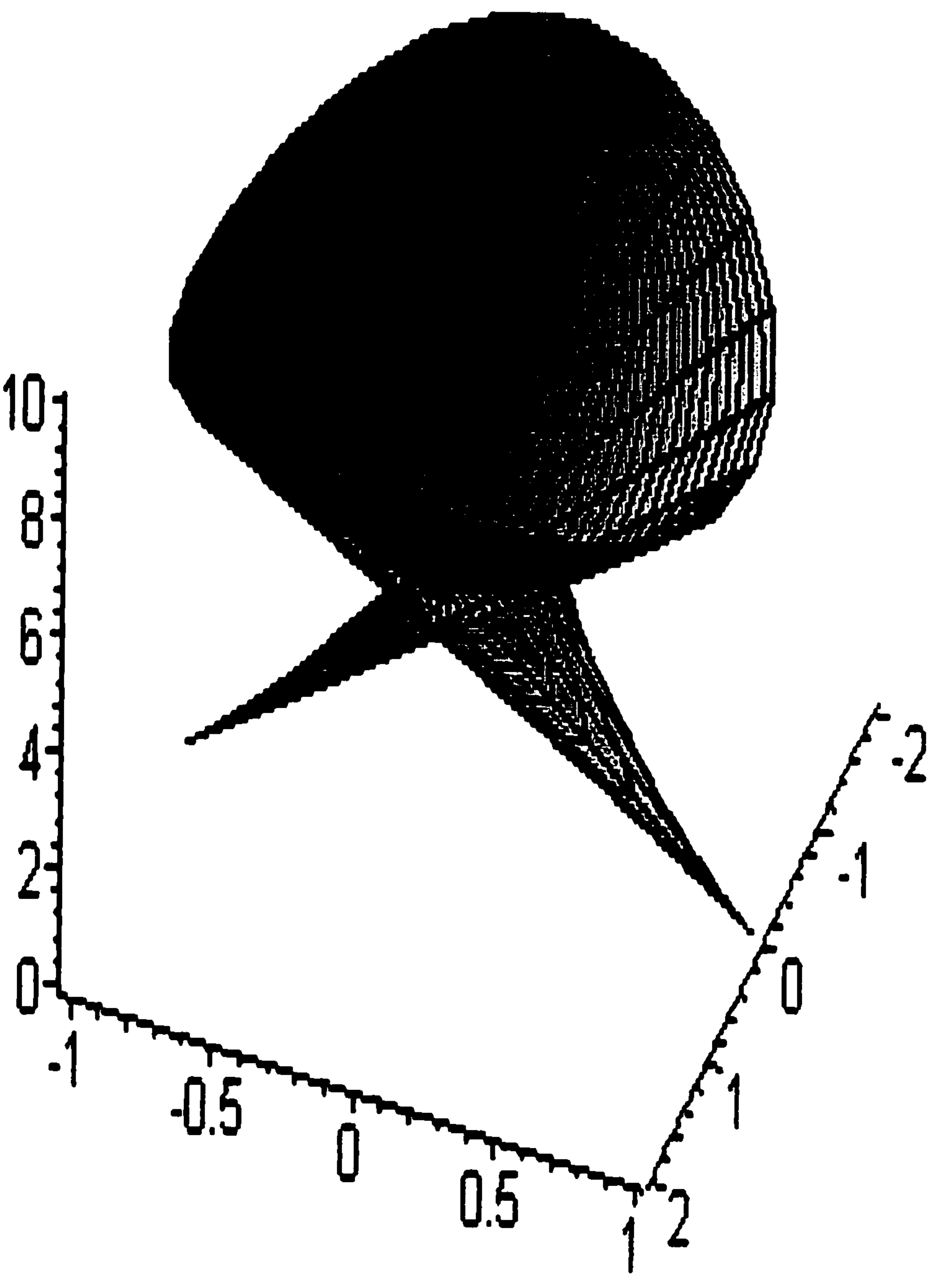

Figure 11.4: Upper Envelope for Paraboloids

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CHAPTER 12

MEASURE OF PLANES SEPARATING A SPHERE AND A CUBE

12.1 Introduction

In this chapter we compute the measure of the set of planes separating a sphere of radius r and a cube with side s distance c apart and sharing an axis of symmetry orthogonal to two opposite faces of the cube.

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12.2 A Convenient Parametrization

Let $(x, y, z) \in R^3$. Without loss of generality assume that the z-axis is the shared axis of symmetry. We parametrize the sphere as

$$S_A(\theta, \phi) = \big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big).$$

Also without loss of generality we can position the cube so that each vertex has either an x-coordinate or a y-coordinate that is zero. Thus we take the vertices on the face of the cube closest to the sphere as

$$\left(\frac{s}{\sqrt{2}}, 0, c+r\right), \left(0, \frac{s}{\sqrt{2}}, c+r\right), \left(-\frac{s}{\sqrt{2}}, 0, c+r\right), \text{ and} \left(0, -\frac{s}{\sqrt{2}}, c+r\right).$$

12.3 Envelope

Lemma 12.3.1 *Let S_A denote the sphere*

$$S_A(\theta, \phi) = \big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big).$$

Let S_B denote the cube with lower vertices

$$\left(\frac{s}{\sqrt{2}}, 0, c+r\right), \left(0, \frac{s}{\sqrt{2}}, c+r\right), \left(-\frac{s}{\sqrt{2}}, 0, c+r\right), \text{ and} \left(0, -\frac{s}{\sqrt{2}}, c+r\right).$$

Then for

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

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(i) the value of ϕ at separating double support points is

$$\phi(\theta) = \cos^{-1}\left(\frac{2r(c+r) + s\cos\theta\sqrt{s^2\cos^2\theta + 2(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta}\right) \text{ and}$$

(ii) the locus of separating double support on the sphere is

$$\theta \to \left(r\cos\theta\frac{-\sqrt{2}rs\cos\theta + (c+r)\sqrt{2s^2\cos^2\theta + 4(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta}.\right.$$

$$r\sin\theta\frac{-\sqrt{2}rs\cos\theta + (c+r)\sqrt{2s^2\cos^2\theta + 4(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta}.$$

$$\left.\frac{2r^2(c+r) + rs\cos\theta\sqrt{s^2\cos^2\theta + 2(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta}\right).$$

Proof. We first compute the normal vector to the sphere

$$N_A(\theta.\phi) = (\cos\theta\sin\phi.\sin\theta\sin\phi.\cos\phi).$$

Next we note that the normal vector of any separating double support plane will be the N_A. the normal vector to the sphere. Thus we take the dot products of N_A with a point on the sphere and with a vertex of the cube to get

$$N_A \cdot S_A = r$$

$$N_A \cdot \left(-\frac{s}{\sqrt{2}}.0.c+r\right) = -\frac{s}{\sqrt{2}}\cos\theta\sin\phi + (c+r)\cos\phi.$$

Then solving

$$N_A \cdot S_A = N_A \cdot \left(-\frac{s}{\sqrt{2}}.0.c+r\right)$$

yields

$$\sin\phi = \frac{-\sqrt{2}rs\cos\theta + (c+r)\sqrt{2s^2\cos^2\theta + 4(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta}.$$

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$$\cos\phi = \frac{2r(c+r) + s\cos\theta\sqrt{s^2\cos^2\theta + 2(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta}\Bigg).$$

This proves (i) and in the equation for S_A replacing $\sin\phi$ and $\cos\phi$ with the above expressions gives the desired result in (ii). □

Theorem 12.3.2 *Let S_A denote the sphere*

$$S_A(\theta,\phi) = \big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big).$$

Let S_B denote the cube with lower vertices

$$\Big(\frac{s}{\sqrt{2}}, 0, c+r\Big), \Big(0, \frac{s}{\sqrt{2}}, c+r\Big), \Big(-\frac{s}{\sqrt{2}}, 0, c+r\Big), \text{ and}\Big(0, -\frac{s}{\sqrt{2}}, c+r\Big).$$

Then

(i) the envelope of separating double support planes is symmetric with respect to rotations of angle $\pi/2$ about the z-axis and contains a planar portion and a non-planar portion.

$$\text{(ii) For } -\frac{\pi}{4} < \theta < \frac{\pi}{4} \text{ and } 0 < t < 1$$

the nonplanar portion of the envelope may be parametrized by

$$S(\theta,t) = \Bigg(\frac{(t-1)s}{\sqrt{2}} + rt\cos\theta\frac{-\sqrt{2}rs\cos\theta + (c+r)\sqrt{2s^2\cos^2\theta + 4(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta},$$

$$rt\sin\theta\frac{-\sqrt{2}rs\cos\theta + (c+r)\sqrt{2s^2\cos^2\theta + 4(c^2+2cr)}}{2(c+r)^2 + s^2\cos^2\theta},$$

$$(1-t)(c+r) + t\frac{2r^2(c+r) + rs\cos\theta\sqrt{2(c^2+2cr) + s^2\cos^2\theta}}{2(c+r)^2 + s^2\cos^2\theta}\Bigg)$$

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(iii) and for $\theta = \pi/4, 0 < p < 1,$ and $0 < q < 1$

the planar portion of the envelope may be parametrized by

$$S(p,q) = \left(\frac{(q-1)ps}{\sqrt{2}} + q\frac{-\sqrt{2}r^2 s + \sqrt{2}r(c+r)\sqrt{s^2+4(c^2+2cr)}}{s^2+4(c+r)^2}, \right.$$
$$\frac{(1-q)(p-1)s}{\sqrt{2}} + q\frac{-\sqrt{2}r^2 s + \sqrt{2}r(c+r)\sqrt{s^2+4(c^2+2cr)}}{s^2+4(c+r)^2},$$
$$\left. (1-q)(c+r) + q\frac{4r^2(c+r) + rs\sqrt{s^2+4(c^2+2cr)}}{s^2+4(c+r)^2} \right).$$

Proof of (i). The symmetry of the envelope follows from the $\pi/2$ rotational symmetry of the original sphere cube pair. The type of curvature of the envelope follows from the type of curvature of the original pair.

Proof of (ii). To get the formula for the nonplanar portion of the envelope we connect the separating double support points of the sphere with the appropriate vertex of the cube. Thus

$$S(\theta, t) = tS(\theta, \phi(\theta)) + (1-t)\left(-\frac{s}{\sqrt{2}}, 0, c+r\right)$$

where $S(\theta, \phi(\theta))$ is the locus of separating double support points on the sphere given by Lemma 12.3.1 above. Then replacing $S(\theta, \phi(\theta))$ with the formula given in the lemma yields the desired result.

Proof of (iii). The planar part of the envelope will be a triangle with two vertices on the cube and one vertex at the point $S_A(\pi/4, \phi(\pi/4))$ on the sphere.

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We first compute

$$S_A(\pi/4, \phi(\pi/4))$$

$$= \left(\frac{-\sqrt{2}r^2 s + \sqrt{2}r(c+r)\sqrt{s^2+4(c^2+2cr)}}{s^2+4(c+r)^2}, \right.$$

$$\frac{-\sqrt{2}r^2 s + \sqrt{2}r(c+r)\sqrt{s^2+4(c^2+2cr)}}{s^2+4(c+r)^2},$$

$$\left. \frac{4r^2(c+r) + rs\sqrt{s^2+4(c^2+2cr)}}{s^2+4(c+r)^2} \right).$$

We next parametrize the appropriate edge of the cube

$$(1-p)\Big(0, -\frac{s}{\sqrt{2}}, c+r\Big) + p\Big(-\frac{s}{\sqrt{2}}, 0, c+r\Big)$$

$$= \Big(-\frac{ps}{\sqrt{2}}, \frac{(p-1)s}{\sqrt{2}}, c+r\Big).$$

We then parametrize the triangle

$$(1-q)\Big(-\frac{ps}{\sqrt{2}}, \frac{(p-1)s}{\sqrt{2}}, c+r\Big) + qS_A(\pi/4, \phi(\pi/4))$$

to get the desired result. □

12.4 Direct Computation of Separating Measure

Theorem 12.4.1 *Let S_A denote the sphere*

$$S_A(\theta, \phi) = \big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big).$$

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Let S_B denote the cube with lower vertices

$$\left(\frac{s}{\sqrt{2}},0,c+r\right),\left(0,\frac{s}{\sqrt{2}},c+r\right),\left(-\frac{s}{\sqrt{2}},0,c+r\right),\ \text{and}\left(0,-\frac{s}{\sqrt{2}},c+r\right).$$

The measure of planes separating the two bodies is

$$\pi(c-r)+2\int_{-\pi/4}^{\pi/4}\left[\frac{2r^2(c+r)+rs\cos\theta\sqrt{2(c^2+2cr)+s^2\cos^2\theta}}{2(c+r)^2+s^2\cos^2\theta}\right.$$
$$\left.-\frac{\sqrt{2}}{2}rs\cos\theta\tan^{-1}\left(\frac{\sqrt{2}}{r}\frac{-rs\cos\theta+(c+r)\sqrt{2(c^2+2cr)+s^2\cos^2\theta}}{2r(c+r)+s\cos\theta\sqrt{2(c^2+2cr)+s^2\cos^2\theta}}\right)\right]\,d\theta.$$

Proof. First determine the z-intercept of the separating double support planes by solving

$$N_A(\theta,\phi(\theta))\cdot S_A(\theta,\phi(\theta))=N_A(\theta,\phi(\theta))\cdot(0,0,z)$$

for z to get

$$r=z\frac{2r^2(c+r)+rs\cos\theta\sqrt{s^2\cos^2\theta+2(c^2+2cr)}}{2(c+r)^2+s^2\cos^2\theta}\Bigg)$$

which implies that

$$z=z(\theta)=\frac{2r(c+r)^2+rs^2\cos^2\theta}{2r(c+r)+s\cos\theta\sqrt{2(c^2+2cr)+s^2\cos^2\theta}}.$$

For $z<z(\theta)$ separating planes will be bounded by tangent planes to the sphere. Thus solving

$$N_A(\theta,\phi)\cdot S_A(\theta,\phi)=N_A(\theta,\phi)\cdot(0,0,z)$$

for ϕ gives

$$\phi=\phi(z)=\cos^{-1}\left(\frac{r}{z}\right)$$

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at points of tangency on the sphere. For $z > z(\theta)$ separating planes will be bounded by support planes to the cube. Thus solving

$$N_A(\theta,\phi)\cdot\left(-\frac{s}{\sqrt{2}},0,c+r\right) = N_A(\theta,\phi)\cdot(0,0,z)$$

for ϕ implies

$$-\frac{rs}{\sqrt{2}}\sin\phi\cos\theta + (c+r)\cos\phi = z\cos\phi$$

which implies

$$\phi = \phi(\theta,z) = \cos^{-1}\left(\frac{rs\cos\theta}{\sqrt{2(c+r-z)^2 + r^2s^2\cos^2\theta}}\right).$$

Thus taking advantage of the $\pi/2$ rotational symmetry the separating measure is

$$4\int_{-\pi/4}^{\pi/4}\int_{r}^{z(u)}\int_{0}^{\cos^{-1}\left(\frac{r}{z}\right)} \cos\phi\sin\phi\; d\phi\; dz\; d\theta$$

$$+4\int_{-\pi/4}^{\pi/4}\int_{z(u)}^{c+r}\int_{0}^{\cos^{-1}\left(\frac{rs\cos\theta}{\sqrt{2(c+r-z)^2+r^2s^2\cos^2\theta}}\right)} \cos\phi\sin\phi\; d\phi\; dz\; d\theta.$$

Then partially evaluating the integrals and simplifying gives the desired result.

□

12.5 Mean Curvature Integrals

Lemma 12.5.1 *Let S_A denote the sphere*

$$S_A(\theta,\phi) = \left(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\right).$$

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Let S_B denote the cube with lower vertices

$$\left(\frac{s}{\sqrt{2}}, 0, c+r\right), \left(0, \frac{s}{\sqrt{2}}, c+r\right), \left(-\frac{s}{\sqrt{2}}, 0, c+r\right), \text{ and} \left(0, -\frac{s}{\sqrt{2}}, c+r\right).$$

Then the total absolute mean curvature over the spherical cap is

$$2\pi r - 4r\int_{-\pi/4}^{\pi/4} \frac{2r(c+r) + s\cos\theta\sqrt{2(c^2+2cr)+s^2\cos^2\theta}}{2(c+r)^2+s^2\cos^2\theta}\, d\theta.$$

Proof. Because of the symmetry of the sphere/cube pair with respect to rotations by angle $\pi/2$ we need only consider one fourth of the spherical cap. The cap will be bounded by points of separating double support. We first determine the angle ϕ at points of separating double support by solving

$$N_A \cdot S_A = N_A \cdot \left(-\frac{s}{\sqrt{2}}, 0, c+r\right)$$

to get

$$\phi(\theta) = \cos^{-1}\left(\frac{2r(c+r) + s\cos\theta\sqrt{s^2\cos^2\theta + 2(c^2+2cr)}}{2(c+r)^2+s^2\cos^2\theta}\right).$$

The absolute mean curvature on a sphere is well known to be $1/r$ and the area element in these angular coordinates is $\sin^2\phi\, d\phi\, d\theta$. Thus, taking advantage of the fourfold symmetry, the integral of mean curvature over the spherical cap is

$$4\int_{-\pi/4}^{\pi/4}\int_0^{\phi(\theta)} r\sin\phi\, d\phi\, d\theta$$

Evaluating and simplifying this integral then gives the desired result. □

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Lemma 12.5.2 *Let S_A denote the sphere*

$$S_A(\theta, \phi) = \left(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\right).$$

Let S_B denote the cube with lower vertices

$$\left(\frac{s}{\sqrt{2}}, 0, c+r\right), \left(0, \frac{s}{\sqrt{2}}, c+r\right), \left(-\frac{s}{\sqrt{2}}, 0, c+r\right), \text{ and} \left(0, -\frac{s}{\sqrt{2}}, c+r\right).$$

Then the total absolute wedge function over the cubical cap is

$$2s\cos^{-1}\left(\frac{4r(c+r) + s\sqrt{4(c^2+2cr)+s^2}}{4(c+r)^2+s^2}\right).$$

Proof. Since the cap in this example has only one face we only need to compute the angle between that face and the envelope. The normal vector to this face is $(0, 0, 1)$. The normal vector to the envelope is

$$N_A(\theta, \phi) = \left(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\right).$$

Thus the normal vector for the separating double support plane of the flat face of the envelope is $N_A(\pi/4, \phi(\pi/4))$ where $\phi(\theta)$ is given in Lemma 12.3.1. Thus the angle between the flat part of the envelope and the face of the cube is

$$\cos^{-1}\left(N_A(\pi/4, \phi(\pi/4)) \cdot (0, 0, 1)\right)$$
$$= \cos^{-1}\left(\frac{4r(c+r) + s\sqrt{4(c^2+2cr)+s^2}}{4(c+r)^2+s^2}\right).$$

The wedge function is half the length of the side times the angle. There are four of them. Thus multiplying the result above by $2s$ gives the desired result.

□

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Conjecture 12.5.3 *Let S_A denote the sphere*

$$S_A(\theta, \phi) = \big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big).$$

Let S_B denote the cube with lower vertices

$$\Big(\frac{s}{\sqrt{2}}, 0, c+r\Big), \Big(0, \frac{s}{\sqrt{2}}, c+r\Big), \Big(-\frac{s}{\sqrt{2}}, 0, c+r\Big), \text{ and}\Big(0, -\frac{s}{\sqrt{2}}, c+r\Big).$$

Then the measure of planes separating the cube and the sphere is equal to the total absolute mean curvature/wedge function over the envelope minus the total absolute mean curvature/wedge function over the caps.

Numerical Evidence. The mean curvature over the envelope was computed for particular values of c, r, and s using the mathematics software package called Mathematica 4 and a slightly updated version of a program written by Alfred Gray for Mathematica 2. See Gray (1993, 302). The integrals were then evaluated using Maple and Simpson's Rule for particular values of the parameters. See Table 12.5 below.

Table 12.1: Computations for a Sphere and a Cube

c	r	s	Separating Measure	Mean Curvature Integral
1	1	$\sqrt{2}$	0.54951627 28	0.5495167 4
3	1	$\sqrt{2}$	4.58428782 4	4.58428782 6

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12.6 Figures

The measure of planes separating a sphere and a cube is an interesting example to study because it is the first example we have considered in which one of the convex bodies was smooth and the other polyhedral. The other pairs we have studied have been either polyhedral or smooth but not mixed. It is also interesting because the separating double support planes do not meet in a point and thus the envelope is not conical as can be seen in Figure 12.1.

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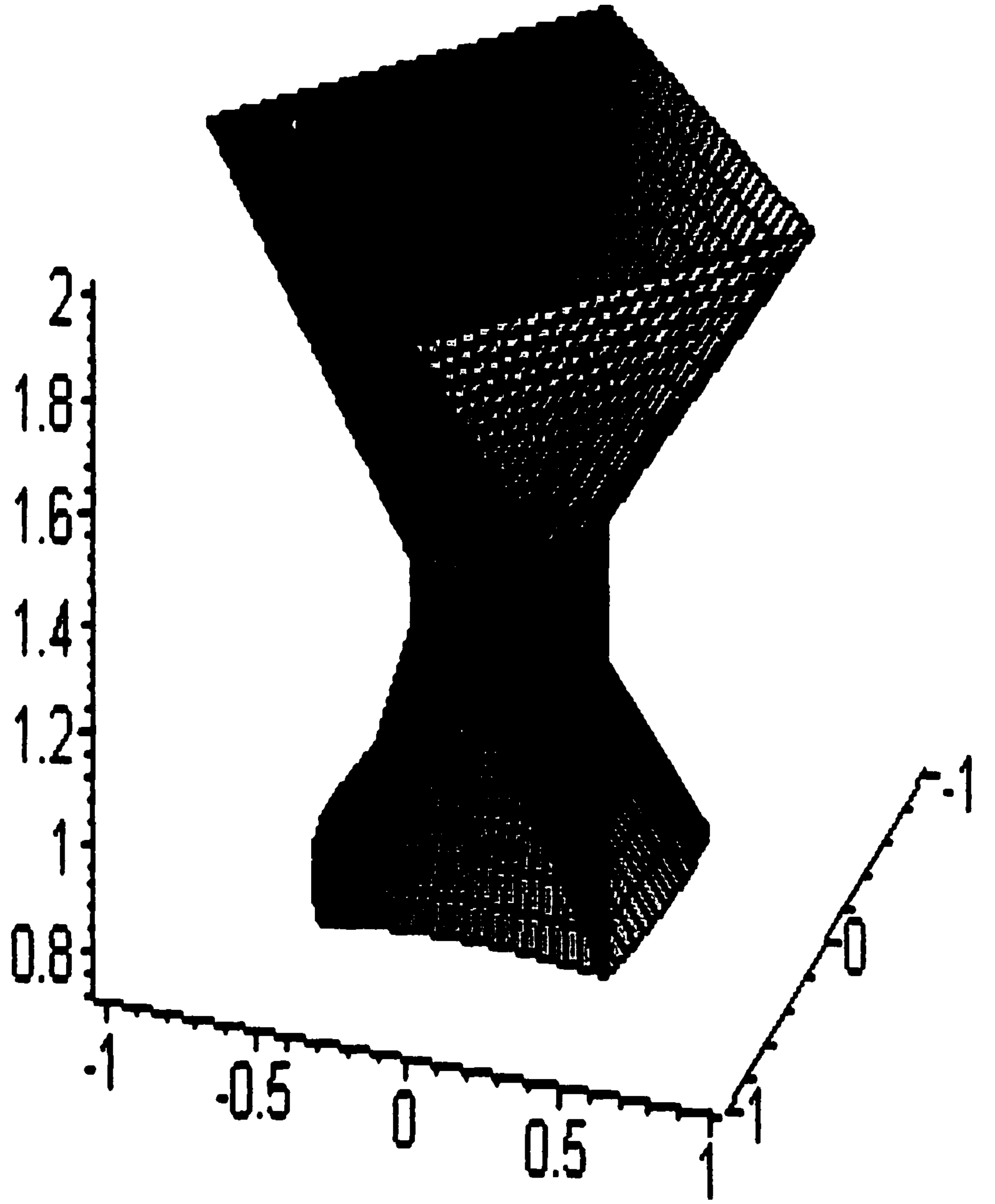

Figure 12.1: Envelope for a Sphere and a Cube

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APPENDIX A

SAMPLE FORTRAN 90 PROGRAMS

A.1 Description of the Programs

The following Fortran 90 programs use Simpson's Rule to approximate integrals derived in Chapter 10 associated with a pair of convex bodies A and B in R^3. Temple graduate student Jian Jun Xu assisted this author by writing a Fortran program that applied Simpson's Rule to the integral of mean curvature over the caps and which could be easily adapted to the other two integrals.

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The boundary of A is a fourth order surface parametrized by $x_1(x, y) = (x, y, -x^4 - y^4)$. The boundary of B is a fourth order surface whose boundary is parametrized by $x_2(x, y) = (-x, -y, x^4 + y^4 + c)$.

In the sample programs below the constant c in the parametrization of the boundary of B is taken to be 7. To compute the integrals for other values of c all occurences of 7 in the programs below were replaced by the desired value of c. To change the number of nodes, the value of $n1$ can be changed in the programs below.

To run the programs, first each program was saved separately to a text file. In that text file each Fortran command started in a new line in column 7 or higher rather than at the beginning of the line. The filename was of the form "filename".f where "filename" was the same as the program name given below. Next each program was compiled on a Unix system with Fortran 90 with a command of the form

f90 -r8 -fast -o "filename" "filename".f

Finally each program was run from the Unix command line by typing

nice "filename" > "filename".out &

The result after running these particular programs is given in Table A.1.

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Table A.1: Computations for a Fourth Order Surface

Absolute Mean Curvature over Caps Part I	3.09462800370188
Absolute Mean Curvature over Caps Part II	9.97020711482128
Caps Total	13.06483511852316
Absolute Mean Curvature over Envelope	29.4188361301735
Difference (Mean Curvature Integral)	16.354001011650 34
Separating Measure	16.354001011650 8

A.2 Separating Measure

This first program directly computes the measure of planes separating the convex bodies A and B. The name of the file is x4y4d7.f

```
      program x4y4d7
c     modified July 5, 2001
c     to compile     f90 -r8 -fast -o filename filename.f
c     to run         nice filename > filename.out &
c     to change distance change lines   program, d=
      parameter n1=72000
      parameter n=2*n1
      implicit real(a-h,o-z)
      dimension x(0:n),y(0:n)
      dimension v(0:n)
      write(6,*)'n,n1:',n,n1
      a=0.
      b=ACOS(-1.)/4.
      c=0.
      d=7.0/2.0
      write(6,*)'a,b',a,b
      h=(b-a)/n1
      do i=0,n
       x(i)=a+i*h/2.
       h1=(d-c)/n1
       do j=0,n
        y(j)=c+h1/2.*j
```

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```
          enddo
          v(i)=0.
          do j=0,n1-1
           v(i)=v(i)+h1/6*(f(x(i),y(2*j))
     /      +4.*f(x(i),y(2*j+1))+f(x(i),y(2*j+2)))
          enddo
         enddo
         value=0.
         do i=0,n1-1
          value=value+h/6.*(v(2*i)+4.*v(2*i+1)+v(2*i+2))
         enddo
         write(6,*)'direct=',value
         stop
         end

         function f(u,z)
         implicit real(a-h,o-z)
         f=128.*SQRT(z)**3/(16.*SQRT(z)**3
     /   +3.*SQRT(3.)*SQRT(SIN(u)**(4./3.)
     /   +COS(u)**(4./3.))**3)
         return
         end
```

A.3 Mean Curvature over the Envelope

This next program computes the total absolute mean curvature over the envelope of separating double support planes of the convex bodies A and B. The name of the file is x4y4ce7.f.

```
      program x4y4e7

c     modified July 5, 2001
c     to compile      f90 -r8 -fast -o filename filename.f
c     to run          nice filename > filename.out
```

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```
C       to change distance change lines  program, b=, f=
        parameter n1=576000
        parameter n=2*n1
        implicit real(a-h,o-z)
        dimension x(0:n)
        write(6,*)'n,n1:',n,n1
        a=0.
        b=(7./12.)**0.25
        write(6,*)'a,b',a,b
        h=(b-a)/n1
        do i=0,n
        x(i)=a+i*h/2.
        enddo
        value=0.
        do i=0,n1-1
         value=value+h/6.*(f(x(2*i))+4.*f(x(2*i+1))+f(x(2*i+2)))
        enddo
        write(6,*)'env=',value
        stop
        end

        function f(u)
        implicit real(a-h,o-z)
        f=32.*u**2*(9.*u**2+4.*7.**2+9.*SQRT((7./6.)-u**4))/
     /  (3.*((7./6.)-u**4)**0.25
     /  *(16.*u**6+1.+16.*(SQRT((7./6.)-u**4)**3)))
        return
        end
```

A.4 Mean Curvature over the Caps

These last two programs compute the total absolute mean curvature over two pieces of the caps of the convex bodies A and B. The names of the files are x4y4c7a.f and x4y4c7b.f

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```
      program x4y4c7a

c     modified July 5, 2001
c     compile with      f90 -r8 -fast -o filename filename.f
c     run with          nice filename > filename.out &
c     distance is indicated in program, b=,c=,and d= lines
      parameter n1=144000
      parameter n=2*n1
      implicit real(a-h,o-z)
      dimension x(0:n),y(0:n)
      dimension v(0:n)
      write(6,*)'n,n1:',n,n1
      a=0.
      b=SQRT(SQRT(7.0/12.0))
      write(6,*)'a,b',a,b
      h=(b-a)/n1
      do i=0,n
       x(i)=a+i*h/2.
       h1=(d(x(i))-c(x(i)))/n1
       do j=0,n
        y(j)=c(x(i))+h1/2.*j
       enddo
       v(i)=0.
       do j=0,n1-1
        v(i)=v(i)+h1/6*(f(x(i),y(2*j))
     /  +4.*f(x(i),y(2*j+1))+f(x(i),y(2*j+2)))
       enddo
      enddo
      value=0.
      do i=0,n1-1
       value=value+h/6.*(v(2*i)+4.*v(2*i+1)+v(2*i+2))
      enddo
      write(6,*)'capa=',value
      stop
      end

      function f(u,v)
      implicit real(a-h,o-z)
      f=96.*(u**2+16.*u**2*v**6+v**2+16.*v**2*u**6)
     / /(1.+16.*u**6+16.*v**6)
```

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```
      return
      end

      function c(x)
      implicit real(a-h,o-z)
      c=SQRT(SQRT(7.0/12.0))
      return
      end

      function d(x)
      implicit real(a-h,o-z)
      d=SQRT(SQRT(7.0/6.0-x**4))
      return
      end




      program x4y4c7b

c     modified July 5, 2001
c     to compile      f90 -r8 -fast -o filename filename.f
c     to run          nice filename > filename.out &
c     to change distance change lines    program, b=
      parameter n1=576000
      parameter n=2*n1
      implicit real(a-h,o-z)
      dimension x(0:n),y(0:n)
      dimension v(0:n)
      write(6,*)'n,n1:',n,n1
      a=0.
      b=SQRT(SQRT(7.0/12.0))
      write(6,*)'a,b',a,b
      h=(b-a)/n1
      do i=0,n
       x(i)=a+i*h/2.
       h1=(d(x(i))-c(x(i)))/n1
       do j=0,n
        y(j)=c(x(i))+h1/2.*j
```

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```
      enddo
      v(i)=0.
      do j=0,n1-1
       v(i)=v(i)+h1/6*(f(x(i),y(2*j))
/      +4.*f(x(i),y(2*j+1))+f(x(i),y(2*j+2)))
      enddo
    enddo
    value=0.
    do i=0,n1-1
      value=value+h/6.*(v(2*i)+4.*v(2*i+1)+v(2*i+2))
    enddo
    write(6,*)'capb=',value
    stop
    end

    function f(u,v)
    implicit real(a-h,o-z)
    f=96.*(u**2+16.*u**2*v**6+v**2+16.*v**2*u**6)
/     /(1.+16.*u**6+16.*v**6)
    return
    end

    function c(x)
    implicit real(a-h,o-z)
    c=0.
    return
    end

    function d(x)
    implicit real(a-h,o-z)
    d=x
    return
    end
```

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APPENDIX B

Q-BASIC PROGRAM

The following QBasic program computes the measure of planes separating two cubes empirically by randomly placing the cube pairs on a grid of parallel planes and computing the proportion of such pairs which are separated by a plane of the grid. The result is then compared to a numerical result obtained from Ambartzumian's formula (1990, 112).

This program is modeled after a similar QBasic program written by Temple faculty member Eric Grinberg which randomly placed line segments on a grid of parallel lines and used that to compute the probability that a randomly placed line segment intersects a line of the grid.

The program contains some lines that were too long to fit on a printed page. Therefore the long lines were printed here as two or more lines. Here

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the continuation of such lines is indicated with an arrow in the margin. To run the program one needs to delete the arrow and restore the long lines.

The printed program here differs slightly in some other nonessential respects from the program that was actually run. Greek letters and symbols used in the PRINT statements are spelled out here. Also some of the comments were shortened in order to fit on the line. In Q-Basic comments follow a single quote and are ignored by the compiler.

Here is the program.

```
    'To run from DOS type    qbasic /run croft605.bas
    'and then press the      Enter     key.

    DECLARE SUB RotationMatrix (sigma, theta, phi, rotate())
    DECLARE SUB SeparationCheck
->                          (theta, phi, cubes(), separation,
->            j1, t1, u1, v1, j2, t2, u2, v2, j3, t3, u3, v3)
    DECLARE SUB ProgressReport (progress)
    CLS
    PRINT "This program computes the probability that a random
->              plane separates"
    PRINT "two fixed disjoint cubes with equal sides s and
->               such that the second"
    PRINT "cube has fixed orientation (sigma,theta,phi) and
->              fixed center (c1,c2,c3) relative"
    PRINT "to the first cube given that the plane hits the
->           fixed sphere centered"
    PRINT "at (c1/2,c2/2,c3/2) with radius c+SQR(3)*s where c
->              is the distance"
    PRINT "the centers of the two cubes."
    GOTO GetInput
intersect:
    CLS
    PRINT
```

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```
    PRINT "The two cubes must not intersect."
    PRINT
GetInput:
    DO
       PRINT
       PRINT "Please type a number to represent the common
->              value s for the lengths"
       PRINT "of the sides of the two cubes, for example 10,
->               and then press enter."
       PRINT "(s must be greater than 0)"
       INPUT s
       LOOP UNTIL (s > 0)
    DO
       PRINT "Please type three numbers separated by commas
->              to represent the"
       PRINT "orientation (sigma,theta,phi) of the second
->              cube in pi radians relative to the"
       PRINT "first cube, for example .25,.25,.25, and then
->              press the enter key."
       PRINT "((sigma,theta,phi) must satisfy 0<=sigma<.5
->              and 0<=theta<1 and 0<=phi<2)"
       INPUT sigma, theta, phi
       LOOP UNTIL (0 <= sigma AND sigma < .5 AND 0 <= theta
->                AND theta < 1 AND 0 <= phi AND phi < 2)
    PRINT "Please type three numbers separated by commas to
->            represent the"
    PRINT "center (c1,c2,c3) of the second cube relative
->           relative to the"
    PRINT "the first cube, for example 0,20,0, and then press
->           the enter key."
    PRINT "The two cubes must not intersect."
    INPUT c1, c2, c3

    'Set up array giving the coordinates of the vertices of
->             the cube pair.
    'The first component tells which cube.
    'The next three components tell which vertex.
    'The last component tells
->               which rectangular coordinate (x,y, or z).
    DIM cubes(2, 2, 2, 2, 3) 'set up array giving initial
    FOR t = 1 TO 2             'coordinates of the first cube
```

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```
      FOR u = 1 TO 2
  FOR v = 1 TO 2
     cubes(1, t, u, v, 1) = (-1) ^ t * s / 2
     cubes(1, t, u, v, 2) = (-1) ^ u * s / 2
     cubes(1, t, u, v, 3) = (-1) ^ v * s / 2
     NEXT v
     NEXT u
     NEXT t
    pi = 3.14159265#
    sigma = sigma * pi       'rotate about the z-axis
    theta = theta * pi       'rotate about the y-axis
    phi = phi * pi           'rotate about the z-axis
    DIM rotate(3, 3)     'rotate then translate 1st cube
->                                        to get 2nd cube
    CALL RotationMatrix(sigma, theta, phi, rotate())
    FOR t = 1 TO 2
       FOR u = 1 TO 2
  FOR v = 1 TO 2
     FOR w = 1 TO 3
cubes(2, t, u, v, w) = 0
   FOR j = 1 TO 3
   cubes(2, t, u, v, w) = cubes(2, t, u, v, w)
->                       + cubes(1, t, u, v, j) * rotate(w, j)
   NEXT j
   NEXT w
   NEXT v
   NEXT u
   NEXT t
    FOR t = 1 TO 2                 'translate the rotation of
       FOR u = 1 TO 2                      'cube1 to get cube2
  FOR v = 1 TO 2
  cubes(2, t, u, v, 1) = c1 + cubes(2, t, u, v, 1)
  cubes(2, t, u, v, 2) = c2 + cubes(2, t, u, v, 2)
  cubes(2, t, u, v, 3) = c3 + cubes(2, t, u, v, 3)
  NEXT v
  NEXT u
  NEXT t
    FOR t = 1 TO 2    'check that the cubes don't intersect
       FOR u = 1 TO 2
  FOR v = 1 TO 2
     count = 0
```

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```
FOR w = 1 TO 3
IF ((-s / 2) <= cubes(2, t, u, v, w)
->                AND cubes(2, t, u,v, w) <= (s / 2))
->                THEN count = count + 1
IF count = 3 THEN GOTO intersect
NEXT w
NEXT v
NEXT u
NEXT t
    DIM axisrotate(3, 3)                'rotate cubes so their
    squ = SQR(c1 ^ 2 + c2 ^ 2 + c3 ^ 2)
->                                      'centers are on z-axis
    sq = SQR(c1 ^ 2 + c2 ^ 2)
     FOR i = 1 TO 3
     FOR j = 1 TO 3
      axisrotate(i, j) = 0
      IF i = j THEN axisrotate(i, j) = 1
     NEXT j
     NEXT i
    IF c1 = 0 AND c2 = 0 THEN GOTO Position
    axisrotate(1, 1) = c1 * c2 / (squ * sq)
    axisrotate(1, 2) = c2 * c3 / (squ * sq)
    axisrotate(1, 3) = -sq / squ
    axisrotate(2, 1) = -c2 / sq
    axisrotate(2, 2) = c1 / sq
    axisrotate(2, 3) = 0
    axisrotate(3, 1) = c1 / squ
    axisrotate(3, 2) = c2 / squ
    axisrotate(3, 3) = c3 / squ
Position:
    DIM newcubes(2, 2, 2, 2, 3)
     FOR t = 1 TO 2
     FOR u = 1 TO 2
     FOR v = 1 TO 2
     FOR w = 1 TO 3
      newcubes(1, t, u, v, w) = 0
      newcubes(2, t, u, v, w) = 0
     FOR j = 1 TO 3
      newcubes(1, t, u, v, w) = newcubes(1, t, u, v, w)
->                 + cubes(1, t, u, v, j) * axisrotate(w, j)
      newcubes(2, t, u, v, w) = newcubes(2, t, u, v, w)
```

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```
->                  + cubes(2, t, u, v, j) * axisrotate(w, j)
     NEXT j
     NEXT w
     NEXT v
     NEXT u
     NEXT t
    SCREEN 12     'set up color graphics screen
    WINDOW (0, 0)-(600, 300) 'set up window to show screen
    LOCATE 1
    PRINT "Here is the cube pair you have chosen.  In the
->              drawing the cube pair"
    PRINT "is rotated so that the center of each cube is on
->             the vertical axis."
    xcent = 300
    zcent = 150
       FOR j = 1 TO 2                          'draw the cubes
  LINE (xcent + newcubes(j, 1, 1, 1, 1),
->            zcent + newcubes(j, 1,1, 1, 3))
->             -(xcent + newcubes(j, 1, 1, 2, 1),
->            zcent + newcubes(j, 1, 1, 2, 3))
  LINE (xcent + newcubes(j, 1, 1, 1, 1),
->            zcent + newcubes(j, 1, 1, 1, 3))
->            -(xcent + newcubes(j, 1, 2, 1, 1),
->            zcent + newcubes(j, 1, 2, 1, 3))
  LINE (xcent + newcubes(j, 1, 1, 1, 1),
->            zcent + newcubes(j, 1, 1, 1, 3))
->            -(xcent + newcubes(j, 2, 1, 1, 1),
->            zcent + newcubes(j, 2, 1, 1, 3))

  LINE (xcent + newcubes(j, 1, 2, 2, 1),
->            zcent + newcubes(j, 1, 2, 2, 3))
->            -(xcent + newcubes(j, 1, 1, 2, 1),
->            zcent + newcubes(j, 1, 1, 2, 3))
  LINE (xcent + newcubes(j, 1, 2, 2, 1),
->            zcent + newcubes(j, 1, 2, 2, 3))
->            -(xcent + newcubes(j, 1, 2, 1, 1),
->            zcent + newcubes(j, 1, 2, 1, 3))
  LINE (xcent + newcubes(j, 1, 2, 2, 1),
->            zcent +newcubes(j, 1, 2, 2, 3))
->            -(xcent + newcubes(j, 2, 2, 2, 1),
->            zcent + newcubes(j, 2, 2, 2, 3))
```

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```
 LINE (xcent + newcubes(j, 2, 2, 1, 1),
->           zcent + newcubes(j, 2, 2, 1, 3))
->           -(xcent + newcubes(j, 2, 2, 2, 1),
->           zcent + newcubes(j, 2, 2, 2, 3))
 LINE (xcent + newcubes(j, 2, 2, 1, 1),
->           zcent + newcubes(j, 2, 2, 1, 3))
->           -(xcent + newcubes(j, 1, 2, 1, 1),
->           zcent + newcubes(j, 1, 2, 1, 3))
 LINE (xcent + newcubes(j, 2, 2, 1, 1),
->           zcent + newcubes(j, 2, 2, 1, 3))
->           -(xcent + newcubes(j, 2, 1, 1, 1),
->           zcent + newcubes(j, 2, 1, 1, 3))

 LINE (xcent + newcubes(j, 2, 1, 2, 1),
->           zcent + newcubes(j, 2, 1, 2, 3))
->           -(xcent + newcubes(j, 2, 2, 2, 1),
->           zcent + newcubes(j, 2, 2, 2, 3))
 LINE (xcent + newcubes(j, 2, 1, 2, 1),
->           zcent + newcubes(j, 2, 1, 2, 3))
->           -(xcent + newcubes(j, 2, 1, 1, 1),
->           zcent + newcubes(j, 2, 1, 1, 3))
 LINE (xcent + newcubes(j, 2, 1, 2, 1),
->           zcent + newcubes(j, 2, 1, 2, 3))
->           -(xcent + newcubes(j, 1, 1, 2, 1),
->           zcent + newcubes(j, 1, 1, 2, 3))
 NEXT j
  LOCATE 25
  measure = 0
  PRINT
  PRINT "Theoretical computation of probability may take
->          several minutes."
  PRINT "Please type 1 to skip
->                          or any other number not to skip."
  PRINT "Do you wish to skip to empirical computation of
->          probability";
  INPUT skip
  IF skip = 1 THEN GOTO empirical

   '-------------------------------------------------------
   CLS
```

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```
   LOCATE 1
   PRINT "The motion-invariant measure of planes separating
->             two cubes is the"
   PRINT "sum of measures of the solitary separating wedges
->             with a minus or"
   PRINT "plus sign depending on whether the needle of the
->             wedge is an"
   PRINT "edge of a cube or not respectively plus the sum of
->           the clustered"
   PRINT "separating wedges such that the vertices of the
->             needle of the wedge"
   PRINT "are from different cubes. The measure of the wedge
->            is half of the"
   PRINT "length of the needle times the size of the angle.
->             See Ambartzumian's"
   PRINT "red book page 114 for analogous formula for convex
->           polyhedrons.  "
   PRINT
   PRINT "The above measure is converted to a probability
->           measure by dividing by"
   PRINT "the measure of planes hitting a sphere containing
->           the two cubes."
   PRINT "This measure is 2 pi d where d is the diameter of
->         the sphere containing"
   PRINT "the two cubes.  See Ambartzumian's red book
->                                             page 122."
   PRINT
   PRINT "We somewhat arbitrarily take d to be c+SQR(3)*s."
   PRINT
   measure = 0
   progress = -1
   FOR j1 = 1 TO 2   'start compute theoretical probability
   FOR t1 = 1 TO 2
   FOR u1 = 1 TO 2   'pick 4 vertices, compute normal vector
   FOR v1 = 1 TO 2
      vertex1 = 1000 * j1 + 100 * t1 + 10 * u1 + v1
   FOR j2 = 1 TO 2
   FOR t2 = 1 TO 2
   FOR u2 = 1 TO 2
   FOR v2 = 1 TO 2
      CALL ProgressReport(progress)
```

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```
        vertex2 = 1000 * j2 + 100 * t2 + 10 * u2 + v2
        IF vertex2 <= vertex1 THEN GOTO nextsecond
        v121 = cubes(j1, t1, u1, v1, 1)
->                              - cubes(j2, t2, u2, v2, 1)
        v122 = cubes(j1, t1, u1, v1, 2)
->                              - cubes(j2, t2, u2, v2, 2)
        v123 = cubes(j1, t1, u1, v1, 3)
->                              - cubes(j2, t2, u2, v2, 3)
        length = SQR(v121 ^ 2 + v122 ^ 2 + v123 ^ 2)
        IF j1 = j2 AND length >= 1.1 * s
->                                     THEN GOTO nextsecond
        collinear = 0
     FOR j3 = 1 TO 2
     FOR t3 = 1 TO 2
     FOR u3 = 1 TO 2
     FOR v3 = 1 TO 2
        CALL ProgressReport(progress)
        vertex3 = 1000 * j3 + 100 * t3 + 10 * u3 + v3
        IF vertex3 = vertex1 THEN GOTO neksthird
        IF vertex3 = vertex2 THEN GOTO neksthird
        v11 = cubes(j1, t1, u1, v1, 1)
->                                 - cubes(j3, t3, u3, v3, 1)
        v12 = cubes(j1, t1, u1, v1, 2)
->                                 - cubes(j3, t3, u3, v3, 2)
        v13 = cubes(j1, t1, u1, v1, 3)
->                                 - cubes(j3, t3, u3, v3, 3)
        v21 = cubes(j2, t2, u2, v2, 1)
->                                 - cubes(j3, t3, u3, v3, 1)
        v22 = cubes(j2, t2, u2, v2, 2)
->                                 - cubes(j3, t3, u3, v3, 2)
        v23 = cubes(j2, t2, u2, v2, 3)
->                                 - cubes(j3, t3, u3, v3, 3)
        l1 = SQR(v11 ^ 2 + v12 ^ 2 + v13 ^ 2)
        l2 = SQR(v21 ^ 2 + v22 ^ 2 + v23 ^ 2)
        IF (l1 + l2 - length) ^ 2 < .000001
->                                        THEN GOTO nextsecond
        ratio = 0
        IF v21 <> 0 THEN ratio = v11 / v21
        IF v22 <> 0 THEN ratio = v12 / v22
        IF v23 <> 0 THEN ratio = v13 / v23
        IF (v11 - ratio * v21) ^ 2 + (v12 - ratio * v22) ^ 2
```

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```
->        + (v13 - ratio * v23) ^ 2 < .000001
->        THEN collinear = 1
       IF j1 = j2 AND collinear = 1 THEN GOTO nextsecond
neksthird:
       NEXT v3
       NEXT u3
       NEXT t3
       NEXT j3
    FOR j3 = 1 TO 2
    FOR t3 = 1 TO 2
    FOR u3 = 1 TO 2
    FOR v3 = 1 TO 2
       CALL ProgressReport(progress)
       vertex3 = 1000 * j3 + 100 * t3 + 10 * u3 + v3
       IF vertex3 = vertex1 THEN GOTO nextthird
       IF vertex3 = vertex2 THEN GOTO nextthird
       v11 = cubes(j1, t1, u1, v1, 1)
->                                 - cubes(j3, t3, u3, v3, 1)
       v12 = cubes(j1, t1, u1, v1, 2)
->                                 - cubes(j3, t3, u3, v3, 2)
       v13 = cubes(j1, t1, u1, v1, 3)
->                                 - cubes(j3, t3, u3, v3, 3)
       v21 = cubes(j2, t2, u2, v2, 1)
->                                 - cubes(j3, t3, u3, v3, 1)
       v22 = cubes(j2, t2, u2, v2, 2)
->                                 - cubes(j3, t3, u3, v3, 2)
       v23 = cubes(j2, t2, u2, v2, 3)
->                                 - cubes(j3, t3, u3, v3, 3)
       ratio = 0
       IF v21 <> 0 THEN ratio = v11 / v21
       IF v22 <> 0 THEN ratio = v12 / v22
       IF v23 <> 0 THEN ratio = v13 / v23
       IF (v11 - ratio * v21) ^ 2 + (v12 - ratio * v22) ^ 2
->        + (v13 - ratio * v23) ^ 2 < .000001
->         THEN GOTO nextthird
       x1 = v12 * v23 - v22 * v13
       x2 = v21 * v13 - v11 * v23    'compute 1st normal
       x3 = v11 * v22 - v21 * v12
       theta = pi / 2
       phi = 0
       IF x3 > 0 THEN theta = ATN(SQR(x1 ^ 2 + x2 ^ 2) / x3)
```

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```
        IF x3 < 0 THEN theta
->                      = pi + ATN(SQR(x1 ^ 2 + x2 ^ 2) / x3)
        IF x1 > 0 THEN phi = ATN(x2 / x1)
        IF x1 = 0 AND x2 > 0 THEN phi = pi / 2
        IF x1 = 0 AND x2 < 0 THEN phi = 3 * pi / 2
        IF x1 < 0 THEN phi = pi + ATN(x2 / x1)
        CALL SeparationCheck(theta, phi, cubes(), separation,
->           j1, t1, u1, v1, j2, t2, u2, v2, j3, t3, u3, v3)
        IF separation = 0 THEN GOTO nextthird
     '------------------------------------------------------
     FOR j4 = 1 TO 2
     FOR t4 = 1 TO 2
     FOR u4 = 1 TO 2
     FOR v4 = 1 TO 2
        vertex4 = 1000 * j4 + 100 * t4 + 10 * u4 + v4
        IF vertex4 = vertex1 THEN GOTO nextfourth
        IF vertex4 = vertex2 THEN GOTO nextfourth
        IF vertex4 <= vertex3 THEN GOTO nextfourth
        CALL ProgressReport(progress)
        v11 = cubes(j1, t1, u1, v1, 1)
->                                  - cubes(j4, t4, u4, v4, 1)
        v12 = cubes(j1, t1, u1, v1, 2)
->                                  - cubes(j4, t4, u4, v4, 2)
        v13 = cubes(j1, t1, u1, v1, 3)
->                                  - cubes(j4, t4, u4, v4, 3)
        v21 = cubes(j2, t2, u2, v2, 1)
->                                  - cubes(j4, t4, u4, v4, 1)
        v22 = cubes(j2, t2, u2, v2, 2)
->                                  - cubes(j4, t4, u4, v4, 2)
        v23 = cubes(j2, t2, u2, v2, 3)
->                                  - cubes(j4, t4, u4, v4, 3)
        ratio = 0
        IF v21 <> 0 THEN ratio = v11 / v21
        IF v22 <> 0 THEN ratio = v12 / v22
        IF v23 <> 0 THEN ratio = v13 / v23
        IF (v11 - ratio * v21) ^ 2 + (v12 - ratio * v22) ^ 2
->         + (v13 - ratio * v23) ^ 2 < .000001
->         THEN GOTO nextfourth
        xx1 = v12 * v23 - v22 * v13
        xx2 = v21 * v13 - v11 * v23
        xx3 = v11 * v22 - v21 * v12
```

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```
        ratio = 0
        IF xx1 <> 0 THEN ratio = x1 / xx1  'get 2nd normal
        IF xx2 <> 0 THEN ratio = x2 / xx2
        IF xx3 <> 0 THEN ratio = x3 / xx3
        IF (x1 - ratio * xx1) ^ 2 + (x2 - ratio * xx2) ^ 2
->         + (x3 - ratio * xx3) ^ 2 < .000001
->          THEN GOTO nextfourth
        theta = pi / 2
        phi = 0
        IF xx3 > 0 THEN theta =
->                            ATN(SQR(xx1 ^ 2 + xx2 ^ 2) / xx3)
        IF xx3 < 0 THEN theta
->                     = pi + ATN(SQR(xx1 ^ 2 + xx2 ^ 2) / xx3)
        IF xx1 > 0 THEN phi = ATN(xx2 / xx1)
        IF xx1 = 0 AND xx2 > 0 THEN phi = pi / 2
        IF xx1 = 0 AND xx2 < 0 THEN phi = 3 * pi / 2
        IF xx1 < 0 THEN phi = pi + ATN(xx2 / xx1)
        CALL SeparationCheck(theta, phi, cubes(), separation,
->           j1, t1, u1, v1, j2, t2, u2, v2, j4, t4, u4, v4)
        IF separation = 0 THEN GOTO nextfourth
        dot = x1 * xx1 + x2 * xx2 + x3 * xx3
        norms = (x1 ^ 2 + x2 ^ 2 + x3 ^ 2) *
->                                 (xx1 ^ 2 + xx2 ^ 2 +xx3 ^ 2)
        kos = ABS(dot) / SQR(norms)
        IF kos > .1 AND j1 = j2 AND j2 = j3 AND j3 = j4
->                   THEN GOTO nextfourth
        schwarz = norms - dot ^ 2
        IF schwarz < 0 THEN schwarz = 0
        opp = SQR(schwarz)
        IF dot <> 0 THEN angle = ABS(ATN(opp / dot))
        IF dot = 0 THEN angle = pi / 2
        IF j1 = j2 THEN sign = -1
        IF j1 <> j2 THEN sign = 1
        measure = measure + sign * angle * length
        'LPRINT progress; vertex1; vertex2; vertex3; vertex4;
->          "length"; length; "angle"; angle; "sign"; sign
        GOTO nextsecond
    '-------------------------------------------------------
nextfourth:
        NEXT v4
        NEXT u4
```

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```
        NEXT t4
        NEXT j4
nextthird:
        NEXT v3
        NEXT u3
        NEXT t3
        NEXT j3
nextsecond:
        NEXT v2
        NEXT u2
        NEXT t2
        NEXT j2
        NEXT v1
        NEXT u1
        NEXT t1
        NEXT j1
    '-------------------------------------------------------
    d = SQR(c1 ^ 2 + c2 ^ 2 + c3 ^ 2) + SQR(3) * s
    denom = 4 * pi * d
    CLS
    PRINT "The standard separating measure ="; measure / 2
    PRINT
    PRINT "The cubes are contained in a sphere of radius";
->                                                     d / 2
    PRINT "Thus we take our theoretical probability to be the
->         conditional"
    PRINT "probability that a plane separates the cubes given
->           that it hits"
    PRINT "this sphere.  Thus the theoretical"
    PRINT
    PRINT "probability  = measure / (4 * pi * r) =";
->                     measure / (4 * pi * d)
    PRINT
    PRINT
empirical:
    DO
       PRINT
       PRINT "Please type a number N to represent the
->                   number of in pairs of cubes"
       PRINT "to be randomly placed on a grid of parallel
->                  planes order to"
```

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```
      PRINT "compute the empirical probability that a
->              plane separates the cubes."
      PRINT "(N must be a positive integer.)
->                                    How many pairs of cubes";
      INPUT it
      LOOP UNTIL (INT(it) > 0 AND INT(it) = it)

   d = SQR(c1 ^ 2 + c2 ^ 2 + c3 ^ 2) + SQR(3) * s
   'd=37.3205081#
   denom = 4 * pi * d    'finish theoretical probability
   'LPRINT "measure"; measure; "denom"; denom
   prob = measure / denom
   BEEP
   BEEP
   BEEP
   '-----------------------------------------------------
   xcenter = 0
   ycenter = 0
   zcenter = 0
   FOR i = 1 TO 2
   FOR j = 1 TO 2
   FOR k = 1 TO 2
   FOR l = 1 TO 2
    xcenter = xcenter + newcubes(i, j, k, l, 1)
    ycenter = ycenter + newcubes(i, j, k, l, 2)
    zcenter = zcenter + newcubes(i, j, k, l, 3)
   NEXT l
   NEXT k
   NEXT j
   NEXT i
   xcenter = xcenter / 16
   ycenter = ycenter / 16
   zcenter = zcenter / 16
   'CLS
   'PRINT xcenter
   'PRINT ycenter
   'PRINT zcenter
   'INPUT anykey
   DIM centcubes(2, 2, 2, 2, 3)
   FOR i = 1 TO 2
   FOR j = 1 TO 2
```

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```
    FOR k = 1 TO 2
    FOR l = 1 TO 2
     centcubes(i, j, k, l, 1) = newcubes(i, j, k, l, 1)
->                                            - xcenter
     centcubes(i, j, k, l, 2) = newcubes(i, j, k, l, 2)
->                                            - ycenter
     centcubes(i, j, k, l, 3) = newcubes(i, j, k, l, 3)
->                                            - zcenter
    NEXT l
    NEXT k
    NEXT j
    NEXT i

    '--------------------------------------------------
    CLS
    SCREEN 12          'set up color graphics screen
    WINDOW (0, 0)-(600, 300) 'set up window for screen
    m = 280 / d              'number of parallel planes
    FOR i = 0 TO m                  'projection of i-th
       LINE (0, d * i)-(650, d * i), 14   'plane in
       NEXT i                       'color 14 (yellow)
    ' CIRCLE (300, 3 * d / 2), d / 2
    ' CIRCLE (300, 3 * d / 2), d
    LOCATE 3          'i=0 is on the bottom of the screen
    PRINT "To see the cubes clearly turn up the brightness
->          and contrast."
    '---------------------------------------------------
    count = 0
    RANDOMIZE TIMER          'seed random number generator
    FOR i = 1 TO it              'throw i-th pair of cubes
       x = RND(i)            'choose coordinates randomly
       y = RND(i) 'RND(i) gives a random number betw 0 & 1
       dx = d
       IF dx > 280 THEN dx = 280
       x = (560 - 2 * dx) * x + dx   'fit to screen
       linenumber = (280 - 2 * d) \ d  'integer divide
       IF linenumber < 1 THEN linenumber = 1
       y = linenumber * d * y + d
       sigma = 2 * pi * RND(i)
       theta = pi * RND(i)      'set up rotation matrix for
       phi = 2 * pi * RND(i)  'random orientation of cubes
```

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```
        CALL RotationMatrix(sigma, theta, phi, rotate())
        DIM cubesr(2, 2, 2, 2, 3)  'find coordinates of the
        FOR t = 1 TO 2            'vertices of rotated cubes
  FOR u = 1 TO 2
    FOR v = 1 TO 2
FOR w = 1 TO 3
   cubesr(1, t, u, v, w) = 0
   cubesr(2, t, u, v, w) = 0
   FOR j = 1 TO 3
   cubesr(1, t, u, v, w)
->                = cubesr(1, t, u, v, w) +
->                  centcubes(1, t, u, v, j) * rotate(w, j)
   cubesr(2, t, u, v, w)
->                = cubesr(2, t, u, v, w) +
->                  centcubes(2, t, u, v, j) * rotate(w, j)
   NEXT j
   NEXT w
   cubesr(1, t, u, v, 1)
->                             = x + cubesr(1, t, u, v, 1)
   cubesr(1, t, u, v, 2)
->                             = y + cubesr(1, t, u, v, 2)
   cubesr(2, t, u, v, 1)
->                             = x + cubesr(2, t, u, v, 1)
   cubesr(2, t, u, v, 2)
->                             = y + cubesr(2, t, u, v, 2)
   NEXT v
   NEXT u
   NEXT t
        '---------------------------------------------------
        max1 = cubesr(1, 1, 1, 1, 2) 'check plane separates
        min1 = max1             'really only concerned with
        max2 = cubesr(2, 1, 1, 1, 2)   'y-coordinates
        min2 = max2     'since grid is parallel to xz-plane
        FOR t = 1 TO 2
  FOR u = 1 TO 2
    FOR v = 1 TO 2
                 IF max1 < cubesr(1, t, u, v, 2)
->                     THEN max1 = cubesr(1, t, u, v, 2)
                 IF min1 > cubesr(1, t, u, v, 2)
->                     THEN min1 = cubesr(1, t, u, v, 2)
   IF max2 < cubesr(2, t, u, v, 2)
```

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```
->                          THEN max2 = cubesr(2, t, u, v, 2)
   IF min2 > cubesr(2, t, u, v, 2)
->                          THEN min2 = cubesr(2, t, u, v, 2)
   NEXT v
   NEXT u
   NEXT t
        KOLOR = 8
        IF INT(min1 / d) > INT(max2 / d) OR
->              INT(max1 / d) < INT(min2 / d) THEN KOLOR = 0
        IF KOLOR = 0 THEN count = count + 1
        IF KOLOR = 0 THEN KOLOR = (count MOD 14) + 9
        IF KOLOR > 15 THEN KOLOR = KOLOR - 15
        '------------------------------------------------
        c1r = (cubesr(1, 1, 1, 1, 1)
->               + cubesr(1, 2, 2, 2, 1)
->               + cubesr(2, 1, 1, 1, 1)
->               + cubesr(2, 2, 2, 2, 1)) / 4
        c2r = (cubesr(1, 1, 1, 1, 2)
->               + cubesr(1, 2, 2, 2, 2)
->               + cubesr(2, 1, 1, 1, 2)
->               + cubesr(2, 2, 2, 2, 2)) / 4
        'CIRCLE(c1r,c2r),d/2,KOLOR
->                                 'circle rotated cube pair
        'CIRCLE(x+c1/2,y+c2/2),d/2,KOLOR
->                                   'circle before rotation
        '------------------------------------------------
        FOR j = 1 TO 2                          'draw the cubes
  LINE (cubesr(j, 1, 1, 1, 1), cubesr(j, 1, 1, 1, 2))-
->            (cubesr(j, 1, 1, 2, 1),
->            cubesr(j, 1, 1, 2, 2)), KOLOR
  LINE (cubesr(j, 1, 1, 1, 1), cubesr(j, 1, 1, 1, 2))-
->            (cubesr(j, 1, 2, 1, 1),
->            cubesr(j, 1, 2, 1, 2)), KOLOR
  LINE (cubesr(j, 1, 1, 1, 1), cubesr(j, 1, 1, 1, 2))-
->            (cubesr(j, 2, 1, 1, 1),
->            cubesr(j, 2, 1, 1, 2)), KOLOR

  LINE (cubesr(j, 1, 2, 2, 1), cubesr(j, 1, 2, 2, 2))-
->            (cubesr(j, 1, 1, 2, 1),
->            cubesr(j, 1, 1, 2, 2)), KOLOR
  LINE (cubesr(j, 1, 2, 2, 1), cubesr(j, 1, 2, 2, 2))-
```

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```
->          (cubesr(j, 1, 2, 1, 1),
->          cubesr(j, 1, 2, 1, 2)), KOLOR
  LINE (cubesr(j, 1, 2, 2, 1), cubesr(j, 1, 2, 2, 2))-
->          (cubesr(j, 2, 2, 2, 1),
->          cubesr(j, 2, 2, 2, 2)), KOLOR

  LINE (cubesr(j, 2, 2, 1, 1), cubesr(j, 2, 2, 1, 2))-
->          (cubesr(j, 2, 2, 2, 1),
->          cubesr(j, 2, 2, 2, 2)), KOLOR
  LINE (cubesr(j, 2, 2, 1, 1), cubesr(j, 2, 2, 1, 2))-
->          (cubesr(j, 1, 2, 1, 1),
->          cubesr(j, 1, 2, 1, 2)), KOLOR
  LINE (cubesr(j, 2, 2, 1, 1), cubesr(j, 2, 2, 1, 2))-
->          (cubesr(j, 2, 1, 1, 1),
->          cubesr(j, 2, 1, 1, 2)), KOLOR

  LINE (cubesr(j, 2, 1, 2, 1), cubesr(j, 2, 1, 2, 2))-
->          (cubesr(j, 2, 2, 2, 1),
->          cubesr(j, 2, 2, 2, 2)), KOLOR
  LINE (cubesr(j, 2, 1, 2, 1), cubesr(j, 2, 1, 2, 2))-
->          (cubesr(j, 2, 1, 1, 1),
->          cubesr(j, 2, 1, 1, 2)), KOLOR
  LINE (cubesr(j, 2, 1, 2, 1), cubesr(j, 2, 1, 2, 2))-
->          (cubesr(j, 1, 1, 2, 1),
->          cubesr(j, 1, 1, 2, 2)), KOLOR
  NEXT j
      '--------------------------------------------------
      LOCATE 1      'summarize results at top of screen
      PRINT count; "separation(s) in "; i; "tries."
      PRINT "Empirical probability =";
->                                count / i; "          "

      FOR j = 1 TO 10
->            'this loop checks 10 times if user pressed key
   IF INKEY$ <> "" THEN STOP
   NEXT j
      NEXT i              'done with the i-th pair
      BEEP
      BEEP
      BEEP
      '-----------------------------------------------
```

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```
      LOCATE 30
      PRINT "Press any key to continue.
->                               Are you ready to continue";
      INPUT anykey
      CLS
      LOCATE 1        'summarize results at top of screen
      PRINT count; "separation(s) in "; it; "tries."
      PRINT "Empirical probability =";
->                                count / it; "          "
      PRINT "Estimated standard error =";
->            SQR((count * (it - count))
->              / (it ^ 3)); "          "
      IF prob <> 0 THEN PRINT "Theoretical probability =";
->                                                     prob

   SUB ProgressReport (progress)
      progress = progress + 1
      percent = (100 * progress) \ 3040
      LOCATE 19
      PRINT "Computing theoretical probability . . .
->            very roughly "; percent; "% completed"
      END SUB

   SUB RotationMatrix (sigma, theta, phi, rotate())
      rotate(1, 1) = COS(phi) * COS(theta) * COS(sigma) -
->                 SIN(phi) * SIN(sigma)
      rotate(1, 2) = -COS(phi) * COS(theta) * SIN(sigma) -
->               SIN(phi) * COS(sigma)
      rotate(1, 3) = -COS(phi) * SIN(theta)
      rotate(2, 1) = SIN(phi) * COS(theta) * COS(sigma) +
->              COS(phi) * SIN(sigma)
      rotate(2, 2) = -SIN(phi) * COS(theta) * SIN(sigma) +
->              COS(phi) * COS(sigma)
      rotate(2, 3) = -SIN(phi) * SIN(theta)
      rotate(3, 1) = SIN(theta) * COS(sigma)
      rotate(3, 2) = -SIN(theta) * SIN(sigma)
      rotate(3, 3) = COS(theta)
      END SUB

   SUB SeparationCheck (theta, phi, cubes(), separation,
->            j1, t1, u1, v1, j2, t2, u2, v2, j3, t3, u3, v3)
```

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```
        'check if the plane separates the cubes
        DIM rotate(3, 3)
        DIM test(2, 2, 2, 2, 3)
        rotate(1, 1) = COS(theta) * COS(phi)
        rotate(1, 2) = COS(theta) * SIN(phi)
        rotate(1, 3) = -SIN(theta)
        rotate(2, 1) = -SIN(phi)
        rotate(2, 2) = COS(phi)
        rotate(2, 3) = 0
        rotate(3, 1) = SIN(theta) * COS(phi)
        rotate(3, 2) = SIN(theta) * SIN(phi)
        rotate(3, 3) = COS(theta)
        count1Max = 0
        count1min = 0
        count2Max = 0
        count2min = 0
        FOR t = 1 TO 2
  FOR u = 1 TO 2
     FOR v = 1 TO 2
FOR w = 1 TO 3
   test(1, t, u, v, w) = 0
   test(2, t, u, v, w) = 0
   FOR j = 1 TO 3
   test(1, t, u, v, w) = test(1, t, u, v, w) +
->                          cubes(1, t, u, v, j) * rotate(w, j)
   test(2, t, u, v, w) = test(2, t, u, v, w) +
->                          cubes(2, t, u, v, j) * rotate(w, j)
   NEXT j
   NEXT w
   NEXT v
   NEXT u
   NEXT t
        zMax = test(j1, t1, u1, v1, 3)
        IF zMax < test(j2, t2, u2, v2, 3)
->               THEN zMax =test(j2, t2, u2, v2, 3)
        IF zMax < test(j3, t3, u3, v3, 3)
->               THEN zMax = test(j3, t3, u3, v3, 3)
        zmin = test(j1, t1, u1, v1, 3)
        IF zmin > test(j2, t2, u2, v2, 3)
->               THEN zmin = test(j2, t2, u2, v2, 3)
        IF zmin > test(j3, t3, u3, v3, 3)
```

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```
->              THEN zmin = test(j3, t3, u3, v3, 3)
        FOR t = 1 TO 2
   FOR u = 1 TO 2
      FOR v = 1 TO 2
    IF test(1, t, u, v, 3) <= zMax
->                           THEN count1Max = count1Max + 1
    IF test(1, t, u, v, 3) >= zmin
->                           THEN count1min = count1min + 1
    IF test(2, t, u, v, 3) <= zMax
->                           THEN count2Max = count2Max + 1
    IF test(2, t, u, v, 3) >= zmin
->                           THEN count2min = count2min + 1
    NEXT v
    NEXT u
    NEXT t
        separation = 0
        IF (((count1Max = 8) AND (count2min = 8))
->          OR ((count1min = 8) AND (count2Max = 8)))
->          THEN separation = 1
        END SUB
```

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