Algebra Extravaganza
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Cohomological properties of certain quiver flag varieties

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Joint work.

This is a joint work in-progress with Brian Allen.

Background.

Preliminary definitions. Work over $\mathbb{C}$.

A **quiver** $Q = (Q_0, Q_1)$ is a finite directed graph, where Q_0 is the set of vertices and Q_1 is the set of arrows.

Let $\beta = (\beta_1, \ldots, \beta_{Q_0}) \in \mathbb{N}_{\geq 0}^{Q_0}$, a **dimension vector**.

A **representation of a quiver** assigns a finite-dimensional vector space to each vertex and a linear map to each arrow.

Let

$$Rep(Q, \beta) := \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(W(\beta_{ta}), W(\beta_{ha})),$$

where $W(\beta_i)$ is a β_i-dimensional complex vector space. It has a natural (change-of-basis) group action by $\mathbb{G}_\beta = \prod_{i \in Q_0} GL_{\beta_i}(\mathbb{C})$.

We say $Rep(Q, \beta)$ is the **representation space** of a quiver.

Filtered representation space.

Let $\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(M)} = \beta$ be a sequence of dimension vectors. Then

$$F^\bullet Rep(Q,\beta) := \{(A_a)_{a\in Q_1} : A_a(W(\gamma^{(i)}_{ta})) \subseteq W(\gamma^{(i)}_{ha}) \text{ for all } 1 \leq i \leq M, a \in Q_1\}.$$

$F^\bullet Rep(Q,\beta)$ is the **filtered representation space** for Q and β.

If $P_i \subseteq GL_{\beta_i}(\mathbb{C})$ is the group of automorphisms preserving the filtration of vector spaces at vertex i, then we define

$$\mathbb{P}_\beta := \prod_{i\in Q_0} P_i,$$

which acts as a change-of-basis on $F^\bullet Rep(Q,\beta)$. So there is a parabolic group action $\mathbb{P}_\beta$ on $F^\bullet Rep(Q,\beta)$.

If we have a complete filtration of vector spaces over each vertex, we will write $\mathbb{B}_\beta$ instead of $\mathbb{P}_\beta$.

Universal quiver flag.

We denote

$$Fl_{\gamma^\bullet}(\beta) := \prod_{i \in Q_0} Fl_{\gamma_i^\bullet}(\beta_i),$$

the product of usual flag varieties on the vector space $W(i)$ parametrizing flags of subspaces

$$0 \subseteq U_i^{(1)} \subseteq U_i^{(2)} \subseteq \ldots \subseteq U_i^{(M)} = W(i), \text{ where } \dim(U_i^{(j)}) = \gamma_i^{(j)}, i \in Q_0.$$

We denote $U^\bullet := \{0 \subseteq U^{(1)} \subseteq U^{(2)} \subseteq \ldots \subseteq U^{(M)}\}$, the set of flags of subspaces.

The **universal quiver flag** is defined as

$$Fl_{\gamma^\bullet}^Q(\beta) = \{(W, U^\bullet) \in Rep(Q, \beta) \times Fl_{\gamma^\bullet}(\beta) : W(a)U_{ta}^{(i)} \subseteq U_{ha}^{(i)} \text{ for each } a \in Q_1, 1 \leq i \leq M\}.$$

Projections from the universal quiver flag.

Also denote the universal quiver flag as $\widetilde{Rep(Q, \beta)}$, where we have natural projections p_1 and p_2 into appropriate factors

$$Rep(Q, \beta) \xleftarrow{p_1} \widetilde{Rep(Q, \beta)} \xrightarrow{p_2} Fl_{\gamma^\bullet}(\beta).$$

Properties about morphisms p_1 and p_2

- p_1 and p_2 are $\mathbb{G}_\beta$-equivariant projections,
- a generic fiber of p_1 over W: a product $U^\bullet$ of flags of vector spaces on $U^{(M)}$ such that $WU^\bullet \subseteq U^\bullet$ (so p_1 is projective),
- a generic fiber of p_2 over a flag: a homogeneous vector bundle, i.e., the set of all W such that $WU^\bullet = U^\bullet$ (so p_2 is flat).

- The fibers of p_1 are called **quiver flag varieties**,
- the fibers of p_2 are called **filtered quiver representations**.

Quiver G-S resolution & Springer resolution.

Fix $Q = (Q_0, Q_1)$, where

$$Q_0 = \{v_0, v_1^1, v_2^1, \ldots, v_{\ell_1}^1, v_1^2, v_2^2, \ldots, v_{\ell_2}^2, \ldots, \\ v_1^i, v_2^i, \ldots, v_{\ell_i}^i, \ldots, v_1^k, v_2^k, \ldots, v_{\ell_k}^k\}$$

and $Q_1 = \{a_0^1, a_1^1, \ldots, a_{\ell_1}^1, a_0^2, a_1^2, \ldots, a_{\ell_2}^2, \ldots, a_0^k, a_1^k, \ldots, a_{\ell_k}^k\}$, satisfying

$$v_0 \xleftarrow{a_0^i} v_1^i \xleftarrow{a_1^i} v_2^i \xleftarrow{a_2^i} v_3^i \xleftarrow{a_3^i} \ldots \xleftarrow{a_{\ell_i-1}^i} v_{\ell_i}^i \xleftarrow{a_{\ell_i}^i} v_0 \text{ for } 1 \le i \le k.$$

Also let $M = 2N$. Fix sequences of dimension vectors $\gamma^{(i)}$ so that

$$\begin{aligned} \gamma^{(2i)} &= (i, i, \ldots, i) \text{ for each } 1 \le i \le N \text{ and} \\ \gamma^{(2i-1)} &= (i-1, i-1, \ldots, i-1) \text{ for each } 1 \le i \le N. \end{aligned} \tag{1}$$

Note that $\beta = (N, N, \ldots, N)$.

Generalized G-S resolution.

With Q and β from the previous slide, $\widetilde{Rep(Q, \beta)}$ is called the **generalized Grothendieck-Springer resolution** corresponding to a complete flag. It will be denoted as $\widetilde{Rep(Q, \beta)}_{\mathrm{G-Spr}}$.

Denote $\mathfrak{b}_{Q_1} := F^{\bullet} Rep(Q, \beta)$.

Proposition

The second projection $\widetilde{Rep(Q, \beta)} \xrightarrow{p_2} Fl_{\gamma^{\bullet}}(\beta)$ is $\mathbb{G}_\beta$-equivariant, inducing an isomorphism $\mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{b}_{Q_1} \cong \widetilde{Rep(Q, \beta)}$.

Proposition

A generic fiber of $\mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{b}_{Q_1} \xrightarrow{p_1} Rep(Q, \beta)$ consists of $(N!)^{Q_0}$ points.

Generalized Springer resolution.

Continue to let $M = 2N$. Fix $\gamma^{(i)}$ so that

$$\begin{aligned} \gamma^{(2i)} &= (i, i, \dots, i) \text{ for each } 1 \leq i \leq N \text{ and} \\ \gamma^{(2i-1)} &= (i-1, i-1, \dots, i-1, i) \text{ for each } 1 \leq i \leq N. \end{aligned} \tag{2}$$

For the 1-Jordan quiver,

$$\widetilde{Rep(Q}, \beta) \cong T^*(GL_N(\mathbb{C})/B) \cong GL_N(\mathbb{C}) \times_B \mathcal{N},$$

the cotangent bundle of the flag variety (the Springer resoln of the nilpotent cone $\mathcal{N}$).

For the general quiver and satisfying (2), $\widetilde{Rep(Q}, \beta)$ a **generalized Springer resolution** of the nilpotent cone $\mathcal{N}_{Q_1}$ in $Rep(Q, \beta)$, where $\mathcal{N}_{Q_1} = \{(W(a_0^1), \dots, W(a_{\ell_k}^k)) : W(a_0^1 \cdots a_{\ell_k}^k) \text{ is nilpotent}\}$.

Resolution of singularities.

Now fix the filtration of vector spaces in (2) w.r.t. the standard basis. Then $F^{\bullet}Rep(Q, \beta)$ is contained in the product of upper triangular matrices. Let

$$\mathfrak{n}_{Q_1} = \mathfrak{b}^{\oplus(Q_1 - 1)} \oplus \mathfrak{n}, \tag{3}$$

where $\mathfrak{n} \cong \mathfrak{gl}_n / \mathfrak{b}^*$, the set of strictly upper triangular matrices.

Proposition

$T^*(\mathbb{G}_\beta / \mathbb{B}_\beta) \cong \mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{n}_{Q_1}$ is a resolution of singularities of $\mathcal{N}_{Q_1}$.

Corollary

The morphism $T^*(\mathbb{G}_\beta / \mathbb{B}_\beta) \xrightarrow{\phi} \mathcal{N}_{Q_1}$ is generically finite.

The morphism $T^*(\mathbb{G}_\beta / \mathbb{B}_\beta) \xrightarrow{\phi} \mathcal{N}_{Q_1}$ is also called **generalized Springer desingularization**.

Diagram of morphisms.

Summary

We have the diagram of morphisms, where $\mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{n}_{Q_1} \hookrightarrow \mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{b}_{Q_1}$ is an inclusion of subbundles:

$$
\begin{array}{ccc}
\mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{n}_{Q_1} & \hookrightarrow & \mathbb{G}_\beta \times_{\mathbb{B}_\beta} \mathfrak{b}_{Q_1} \\
\phi \big\downarrow & & \big\downarrow p_1 . \\
\mathcal{N}_{Q_1} & \hookrightarrow & Rep(Q, \beta)
\end{array}
$$

Weights of quiver flag varieties.

For a fixed $\mathbb{G}_\beta$, let $\mathbb{T}_\beta = \prod_{i \in Q_0} T_i$, where T_i is the maximal torus consisting of diagonal matrices in $GL_{\beta_i}(\mathbb{C})$ for each $i \in Q_0$. Let $\mathrm{diag}(t_{i1}, t_{i2}, \ldots, t_{iN}) \in T_i$ for each $i \in Q_0$. Denote $X(\mathbb{T}_\beta)$ to be the character group of $\mathbb{T}_\beta$, where each character $\epsilon_{i,j} \in X(\mathbb{T}_\beta)$ is defined as

$$\epsilon_{i,j}(t_{v_0,1}, \ldots, t_{v_0,N}, \ldots, t_{i,1}, \ldots, t_{i,N}, \ldots, t_{v^k_{\ell_k},1}, \ldots, t_{v^k_{\ell_k},N}) = t_{i,j}$$

for each $i \in Q_0$, $1 \leq j \leq N$.

Then $\{\epsilon_{ij}\}_{i \in Q_0, 1 \leq j \leq N}$ forms a standard basis for $X(\mathbb{T}_\beta)$, which gives an isomorphism of weight lattices $X(\mathbb{T}_\beta) \cong \mathbb{Z}^{|Q_0|N}$.

Dominant weights of $X(\mathbb{T}_\beta)$ are defined to be

$$X^+(\mathbb{T}_\beta) \cong \{\vec{\lambda} = (\lambda^{v_0}, \lambda^{v^1_1}, \lambda^{v^1_2}, \ldots, \lambda^{v^k_{\ell_k}}) \in \mathbb{Z}^{|Q_0|N} : \\ \lambda^i = (\lambda_{i,1} \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,N})\}.$$

Cohomology of line bundles.

Lemma

Let $\vec{\kappa} = \sum_{j=0}^{N} (\epsilon_{v_0,j} - \epsilon_{v_{\ell_k}^k,j})$. Then

$$K(\mathbb{G}_\beta \times_{\mathbb{B}_\beta} (\mathfrak{n}_{Q_1} \times \mathbb{C}_{\vec{\lambda}})) \simeq \mathcal{L}_{\mathbb{G}_\beta \times_{\mathbb{B}_\beta} (\mathfrak{n}_{Q_1} \times \mathbb{C}_{\vec{\lambda}})}(\mathbb{C}_{-\vec{\lambda}-\vec{\kappa}}), \quad (4)$$

where $K(\mathbb{G}_\beta \times_{\mathbb{B}_\beta} (\mathfrak{n}_{Q_1} \times \mathbb{C}_{\vec{\lambda}}))$ is the canonical line bundle of $\mathbb{G}_\beta \times_{\mathbb{B}_\beta} (\mathfrak{n}_{Q_1} \times \mathbb{C}_{\vec{\lambda}})$.

Proposition

Given $\vec{\lambda} \in X^+(\mathbb{T}_\beta)$,

$$H^0(\mathbb{G}_\beta/\mathbb{B}_\beta, \mathcal{L}_{\mathbb{G}_\beta/\mathbb{B}_\beta}(\mathbb{C}_{-\vec{\lambda}})) \cong V^*_{\vec{\lambda}} = V^*_{\lambda^{v_0}} \otimes V^*_{\lambda^{v_1^1}} \otimes \cdots \otimes V^*_{\lambda^{v_{\ell_k}^k}},$$

where $V^*_{\lambda^i}$ is the dual of the $GL_N(\mathbb{C})$-irreducible representation V_{λ^i} with highest weight λ^i.

Higher cohomology.

Theorem

We have

$$H^i(T^*(\mathbb{G}_\beta/\mathbb{B}_\beta), \mathcal{L}_{T^*(\mathbb{G}_\beta/\mathbb{B}_\beta)}(\mathbb{C}_{-\vec{\lambda}})) = 0$$

for any $\vec{\lambda}$ and for all $i > 0$.

Theorem

Given a generalized Grothendieck-Springer resolution $\widetilde{Rep(Q,\beta)}_{\mathrm{G-Spr}}$, we have

$$H^i(\widetilde{Rep(Q,\beta)}_{\mathrm{G-Spr}}, \mathcal{L}_{\widetilde{Rep(Q,\beta)}_{\mathrm{G-Spr}}}(\mathbb{C}_{-\vec{\lambda}})) = 0$$

for any $\vec{\lambda}$ and for all $i > 0$.

Flag Hilbert schemes.

Why are flag Hilbert schemes interesting?

- The adjointness of a monoidal functor from the symmetric monoidal category $\mathcal{C}oh_{\mathrm{FHilb}^n(\mathbb{C})}$ of coherent sheaves on the flag Hilbert scheme $\mathrm{FHilb}^n(\mathbb{C})$ to the non-symmetric monoidal category $\mathbb{S}Bim_n$ of Soergel bimodules produces a 1-1 correspondence between the Euler characteristic of a sheaf on the flag Hilbert scheme with the Hochschild homology of a braid (Gorsky-Negut-Rasmussen, 2016).
- Flag Hilbert schemes are interesting in their own right:
  - birationality of $\mathrm{FHilb}^n(\mathfrak{X})$,
  - singular locus of $\mathrm{FHilb}^n(\mathfrak{X})$,
  - irreducible components of $\mathrm{FHilb}^n(\mathfrak{X})$,
  - Hilbert function of $\mathrm{FHilb}^n(\mathfrak{X})$.

Flag Hilbert schemes.

Constructions

The **flag Hilbert scheme** on $\mathbb{C}$ or $\mathbb{C}^2$ parametrizes full flags of ideals:

$$\mathrm{FHilb}^n(\mathbb{C}) := \{I_n \subseteq \ldots \subseteq I_1 \subseteq I_0 = \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I_i = i, \text{ ideals supported on } y = 0\},$$

$$\mathrm{FHilb}^n(\mathbb{C}^2) := \{I_n \subseteq \ldots \subseteq I_1 \subseteq I_0 = \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I_i = i\}.$$

ADHM description

Let B be lower triangular matrices in $GL_n(\mathbb{C})$, $\mathfrak{b} = \mathrm{Lie}(B)$, and let $\mathfrak{n}$ be nilpotent matrices in $\mathfrak{b}$. Then

$$\mathrm{FHilb}^n(\mathbb{C}) = \{(X,Y,v) \in \mathfrak{b} \times \mathfrak{n} \times \mathbb{C}^n : [X,Y] = 0, X^aY^bv \text{ span } \mathbb{C}^n\}/B,$$

$$\mathrm{FHilb}^n(\mathbb{C}^2) = \{(X,Y,v) \in \mathfrak{b} \times \mathfrak{b} \times \mathbb{C}^n : [X,Y] = 0, X^aY^bv \text{ span } \mathbb{C}^n\}/B.$$

Facts about flag Hilbert schemes.

Facts

- We have $\mathrm{FHilb}^n(\mathbb{C}^2) \twoheadrightarrow \mathrm{Hilb}^n(\mathbb{C}^2)$, sending

$$(I_n \subseteq \ldots \subseteq I_1 \subseteq I_0) \mapsto I_n.$$

- We have $\mathrm{FHilb}^n(\mathbb{C}^2) \twoheadrightarrow \mathbb{C}^{2n}$, sending

$$(I_n \subseteq \ldots \subseteq I_1 \subseteq I_0) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n),$$

  where $(x_j, y_j) = \mathrm{supp}(I_{j-1}/I_j)$.
- $\mathrm{FHilb}^n(\mathbb{C}^2)$ is a closed subscheme of $\mathrm{Hilb}^n(\mathbb{C}^2) \times \mathrm{Hilb}^{n-1}(\mathbb{C}^2) \times \cdots \times \mathrm{Hilb}^1(\mathbb{C}^2) \times \mathrm{Hilb}^0(\mathbb{C}^2)$.
- $\mathrm{FHilb}^n(\mathbb{C})$, $\mathrm{FHilb}^n(\mathbb{C}^2)$ are singular for $n \gg 0$, reducible, their dimensions $\gg$ expected dimensions.

Filtered quiver variety.

Let B be upper triangular matrices in $GL_n(\mathbb{C})$ and let $\mathfrak{b} = \mathrm{Lie}(B)$. Identify $T^*(\mathfrak{b} \times \mathbb{C}^n) \cong \mathfrak{b} \times \mathfrak{b}^* \times \mathbb{C}^n \times (\mathbb{C}^n)^*$. Consider the moment map

$$T^*(\mathfrak{b} \times \mathbb{C}^n) \xrightarrow{\mu_B} \mathfrak{b}^* \cong \mathfrak{gl}_n/\mathfrak{n}^+, \text{ where } (X, Y, v, w) \mapsto [X, Y] + vw.$$

We define

$$\mu_B^{-1}(0)/B := \{(X, Y, v, w) \in \mathfrak{b} \times \mathfrak{b}^* \times \mathbb{C}^n \times (\mathbb{C}^n)^* : [X, Y] + vw = 0, \\ X^a Y^b v \text{ span } \mathbb{C}^n\}/B.$$

Conjecture

There is a birational map $\mu_B^{-1}(0)/B \dashrightarrow \mathrm{FHilb}^n(\mathbb{C}^2)$.

Thank you.
Questions?