Algebras and varieties

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Algebra Extravaganza, July, 2017, Temple University

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Every finite dimensional K-algebra, K algebraically closed, is Morita equivalent to some $K\mathcal{Q}/I$ with $J^N \subseteq I \subseteq J^2$, where $J = \langle \text{arrows} \rangle$

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9. There is a simple algorithm to determine if a monomial algebra is quasi-hereditary.(G.-Schroll, 2017)

Main ideas

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Open questions: (1) (Auslander) Find a categorical description of a monomial algebra.
(2) Given $K\mathcal{Q}/I$, find a criterion to determine if $K\mathcal{Q}/I$ is isomorphic to a monomial algebra.

We recall some well-known definitions.
Let $P_1, \ldots, P_m$ be the indecomposable projective modules $v\Lambda$.
Cartan matrix C_Λ **of** Λ:

$$C_\Lambda = (c_{i,j})$$

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Theorem (Eilenberg)

A a finite dimensional with gldim$(A) < \infty \Rightarrow det(C_A) = \pm 1$.

Conjecture (Cartan determinant conjecture)

A a finite dimensional with gldim$(A) < \infty \Rightarrow det(C_A) = +1$.

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7. If A_{Mon} is quasi-hereditary, then so is A. (G.-Schroll)

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Theorem (G.-Hille-Schroll)

There is an affine algebraic variety $\mathcal{V}(\mathcal{T})$ whose points are in one-to-one correspondence with the algebras Λ such that A is the associated monomial algebra of Λ. The monomial algebra A corresponds to the point $(\mathbf{0}) = (0, 0, \ldots, 0) \in \mathcal{V}(\mathcal{T})$.

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It's time to define the associated monomial algebra.

A brief journey into the world of Gröbner bases

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1. if $p \succ q$ then $pr \succ qr$, if both $\neq 0$.
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If $x = \sum_{p \in \mathcal{B}} \alpha_p p \in K\mathcal{Q}$, then

$$\mathrm{Tip}(x) = \text{ largest } p \text{ ocurring in } x.$$

If $X \subseteq K\mathcal{Q}$, then

$$\mathrm{Tip}(X) = \{\mathrm{Tip}(x) \mid x \in X\}$$

$$\mathrm{Nontip}(X) = \mathcal{B} \setminus \mathrm{Tip}(X).$$

Gröbner bases

Let I be an ideal in KQ. A set of elements $\mathcal{G}$ in I is a Gröbner basis for I with respect to $\succ$ if

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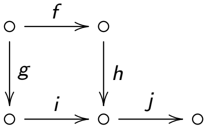
$\langle \text{Tip}(I) \rangle$ is called the associated monomial ideal of I and $K\mathcal{Q}/\langle \text{Tip}(I) \rangle$ is the associated monomial algebra of $K\mathcal{Q}/I$. We sometimes write I_{Mon} for $\langle \text{Tip}(I) \rangle$ and $(K\mathcal{Q}/I)_{Mon}$ for $K\mathcal{Q}/\langle \text{Tip}(I) \rangle = K\mathcal{Q}/I_{Mon}$.

Example

$\mathcal{Q}$ be the quiver



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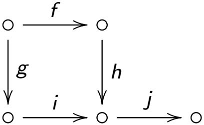
$\mathcal{Q}$ be the quiver 

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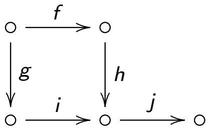

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There is a special Gröbner basis for an ideal that is unique.

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For each $t \in \mathcal{T}$, by the Fundamental Lemma, there exist unique $g_t \in I$ and $n_t \in \text{Span}_K(\mathcal{N})$, such that

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Setting $\mathcal{G} = \{g_t \mid t \in \mathcal{T}\}$, we have:

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$\mathcal{G}$ is called the reduced Gröbner basis of I with respect to $\succ$ and is unique.

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Write elements of $\mathcal{A}$ as tuples $(\mathbf{c}) = (c_{t,n})$ where $t \in \mathcal{T}$, $n \in \mathcal{N}_t$, all but a finite number of entries are nonzero.

Given $(\mathbf{c}) = (c_{t,n}) \in \mathcal{A}$, let H be the ideal generated by

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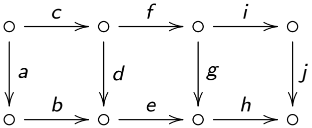
Theorem (Bergman, G, Mora)

Keeping the above notation, $\mathcal{H}$ is the reduced Gröbner basis of H if and only if all overlap relations completely reduce to 0.

Example

Let $\mathcal{Q}$ be



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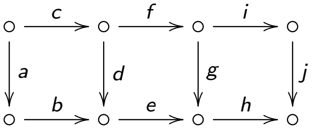
Let $\mathcal{Q}$ be 

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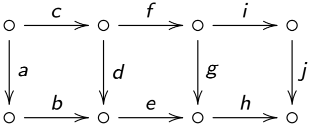
Let $\mathcal{Q}$ be 

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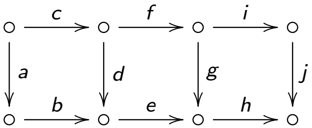

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Thus $\mathcal{V}(\mathcal{T})$ is the zero set of the ideal (XY, YZ) in $K[X, Y, Z]$.

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Fact: If I can generated by weight homogeneous elements, then the reduced Gröbner basis is composed of weight homogeneous elements.

Note that if $\mathcal{T}$ is a set of paths, then $K\mathcal{Q}/\langle\mathcal{T}\rangle$ has an induced weight grading (for any weight function).

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Given a Γ-graded K-algebra $\Lambda = K\mathcal{Q}/I$, where I generated by weight homogeneous elements for some weight function W, there is some set of paths $\mathcal{T}$ such that Λ corresponds to a point in $\mathcal{V}^W(\mathcal{T})$.

Strong Koszul algebras; (G. 2016)

Recall that $\Lambda = K\mathcal{Q}/I$ is a Koszul algebra if the Ext-algebra $\mathrm{Ext}^*_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$ is generated in degrees 0 and 1, where $\mathbf{r} = J/I$.

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We say $\Lambda = K\mathcal{Q}/I$ is a strong Koszul algebra if it has a quadratic Gröbner basis.

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Obtain equations in 13 variables, some of which are degree 4.

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Again we get an affine algebraic subvariety of $\mathcal{V}$ (that need not contain $(\mathbf{0})$).

Admissible ideals

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In general, if $\Lambda'_{mon} = \Lambda$ with $\Lambda' = KQ/I'$, I' need not be admissible.

Proposition

$\mathcal{V}^{ad}$, the algebras in $\mathcal{V}$ defined by admissible ideals is an algebraic subvariety.

Quasi-hereditary algebras (G-Schroll)

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$0 = L_0 \subset L_1 \subset \cdots \subset L_m = \Lambda$ is called a heredity chain.

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Proposition

Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional K-algebra with I a monomial ideal with minimal set ρ of generators of paths. Let v be a vertex of $\mathcal{Q}$. Then $\Lambda v \Lambda$ is a heredity ideal if and only for all $p \in \rho$, v is not internal to p.

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General case

Theorem

Let $\Lambda = K\mathcal{Q}/I$ with $J^N \subseteq I \subseteq J^2$. If Λ_{Mon} is quasi-hereditary, then so is Λ.

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This is necessary and sufficient for $KQeKQ$ to be a heredity ideal in KQ.