Discriminants of Taft algebra smash products & applications

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<u>Setup</u>

$\Bbbk = \overline{\Bbbk}$ field, char $\Bbbk = 0$.

The discriminant is a important algebra invariant, adapted to the nc setting by Ceken, Palmieri, Wang, & Zhang [CPWZ].

<u>Defn</u> Let A be f.g. free over a central subalg C of rank w. The <u>regular trace</u> is defined as the composition

$$tr := tr_{reg} = A \xrightarrow{lm} M_n(C) \xrightarrow{tr_{int}} C.$$

Let $Z = \{z_i\}$ be a finite basis of A over C. The discriminant of A over C is defined to be

$$d(A/C) :=_{C^\times} \det(tr(z_i z_j))_{w \times w} \in C.$$

<u>Key Theorems</u>

① [CPWZ] If $\phi \in \operatorname{Aut} A$ s.t. $\phi(C) = C$, then $\phi(d(A/C)) =_{C^\times} d(A/C)$.

② [Brown, Yakimov] If A is PI prime affine, then

$$\mathcal{V}(d(A/Z(A))) = \operatorname{MaxSpec}(Z(A)) \setminus \mathcal{A}(A),$$

where $\mathcal{A}(A)$ is the Azumaya locus of A.

<u>Motivation</u>

Thm [G, Kirkman, Moore] A an alg, $G \curvearrowright A$ finite grp acting as au[illegible] s.t. no non-id elt of G is inner. If A is f.g. free over a subal[illegible] $C \subseteq Z(A)^G$, then $A \# G$ is f.g. free over C and

$$d(A\#G/C) =_{C^\times} d(A/C)^{|G|}.$$

Q: If we replace G w/ a Hopf alg H and consider the smash product $A \# H$, can we compute the following:

a) $Z(A\#H)$

b) $d(A\#H / Z(A\#H))$

c) $\mathrm{Aut}(A\#H)$

d) ~~Aut~~ $\mathcal{A}(A\#H)$

As a first step we consider H the n^{th} Taft algebra acting on certain quantum algebras.

<u>Taft algebras and actions</u>

<u>Defn</u> Let $n \geq 2$ and λ a prim n^{th} root of unity. The n^{th} Taft algebra is defined to be $H_n(\lambda) = \Bbbk\langle g, x : g^n = 1, x^n = 0, xg = \lambda gx\rangle$. The coalgebra structure of $H_n(\lambda)$ is given by

$$\Delta(g) = g \otimes g \qquad \varepsilon(g) = 1$$

$$\Delta(x) = g \otimes x \qquad \varepsilon(x) = 0$$

We consider actions of $H_n(\lambda)$ on

a) Quantum planes $\Bbbk_\mu[u,v] = \Bbbk\langle u, v : uv = \mu vu\rangle$

b) Quantum Weyl algebras $A_1^\mu = \Bbbk\langle u, v : uv = \mu vu + 1\rangle$

In both cases the action is given by (up to chg of variable)

$$g(u) = \mu u \qquad g(v) = \lambda\mu v \qquad x(u) = 0 \qquad x(v) = \eta u, \quad \eta \in \Bbbk^\times$$

- Can further chg var to get $\eta = 1$
- In general need $|\mu| \,\big|\, |\lambda|$. In case of A_1^μ need $\lambda = \mu^{-2}$
- These both extend the standard action of $H_n(\lambda)$ on $\Bbbk[u,v]$.

We also construct actions on

c) 2×2 quantum matrices

d) a certain quantum 3-space

Thm [GWZ] Let A be an inner faithful ($\exists\, y \in A$ s.t. $x(y) \neq 0$) $H = H_n(\lambda)$-module algebra that is a domain. Suppose for $0 < i < n$, g^i is not inner when restricted to $A^{\langle x \rangle}$. Then $Z(A \# H) = Z(A) \cap A^H$.

As a consequence we obtain in cases (a) & (b) that

$$Z(A \# H) = A^H = \Bbbk[u^m, v^n] \quad \text{where} \quad m = |\mu|.$$

By a result of Bergen, $A \# H$ is prime iff $m = n$ and so we restrict to that case henceforth.

Note Thm also applies to case (c) above but not case (d). However, the result still holds.

Discriminants

As before, $H = H_n(\lambda)$ and $A = \Bbbk_\mu[u,v]$ or A_1^n w/ $n = |\mu| = |\lambda| \geq 2$ and in case $A = A_1^n$ we have n odd

Strategy

1. Recognize $\quad A \# H \cong \dfrac{A[x;\tau,\delta][g;\phi]}{(g^n - 1,\, x^n)}$

2. * Compute discriminant of $A[x;\tau,\delta]$ over $\Bbbk[u^n, v^n, x^n]$ using Poisson techniques [Nguyen, Trampel, Yakimov].

3. Extend to disc of $A[x;\tau,\delta][g;\phi]$ using Ore ext techniques [GKM].

4. Factor to get $d(A\#H / Z(A\#H))$ [CPWZ]

Defn Let R be a $\Bbbk[q^{\pm1}]$ algebra. The <u>specialization</u> of R at $\varepsilon \in \Bbbk^\times$ is defined as $R_\varepsilon := R/(q-\varepsilon)$.

<u>Thm</u> [Brown, Gordon] The canonical projection $\sigma: R \to R_\varepsilon$ induces a Poisson structure on $Z(R_\varepsilon)$ via

$$\{\sigma(x_i), \sigma(x_j)\} = \sigma\left(\frac{x_i x_j - x_j x_i}{q-\varepsilon}\right) \qquad x_i, x_j \in \sigma^{-1}(Z(R_\varepsilon))$$

(Def is indep of choices of preimages).

<u>Thm</u> [NTY] The discriminant of R_ε over $Z(R_\varepsilon)$ is the product of certain poisson ~~poisson~~ prime elements of $Z(R_\varepsilon)$.

In prev work these P. prime elts were determined using Poisson geometry. Another method can be to apply the theory of Poisson Ore extensions [Oh].

$$P[z;\alpha,\beta]: \quad \{z,p\} = \alpha(p)z + \beta(p) \qquad \forall\ p \in P.$$

Let A be a $\Bbbk[q^{\pm1}]$ algebra and (τ,δ) a q-skew extension $(\tau\delta = q\delta\tau)$.

- $(A[t;\tau,\delta])_\varepsilon = A_\varepsilon[t;\tau,\delta]$
- Let B_ε be a central subalg of A_ε and $m = \tau|_{B_\varepsilon}$. The induced P. structure on $B_\varepsilon[t^m]$ is a P. Ore ext of the induced structure on B_ε.

<u>upshot</u>

$A = \Bbbk[q^{\pm1}]\langle u,v : uv - qvu - K\rangle$, $K = 0$ or 1, then $A_\mu = \Bbbk_\mu[u,v]$ or A_1^μ.

Set $R = A[x;\tau,\delta]$ w/ $\tau(u) = qu$, $\tau(v) = q^{K+1}v$, $\delta(u) = 0$, $\delta(v) = u$

where $\lambda = \mu^K$. Set $C_\mu = \Bbbk[\underset{z_1}{u^n}, \underset{z_2}{v^n}, \underset{z_3}{x^n}]$, $n = |\mu|$.

<u>Lem</u> Let $\alpha = n^2(n-1)$, then

$$d(R_m / C_m) =_{\Bbbk^\times} \begin{cases} z_1^\alpha (z_2 z_3 + * z_1)^\alpha \\ (z_1 z_2 z_3 + * z_1^2 + * z_3)^\alpha \end{cases}$$ $* =$ certain scalars.

<u>Thm</u> [GWZ] Let $A = \Bbbk_m[u,v]$ or A_1^m w/ $n = |m|$ and $H = H_n(\lambda)$.

Then $d(A \# H / A^H) =_{\Bbbk^\times} u^{2n^4(n-1)}$

<u>Applications</u> A, H as usual

① Let $n = 2$. Set $S = A \# H$, and

$$r\mathrm{Aut}(S) = \{ \phi \in \mathrm{Aut}(S) : \underset{\varepsilon = \pm 1}{\phi(1 \# g) = \varepsilon(1 \# g)}, \ \underset{\gamma \in \Bbbk^\times}{\phi(1 \# x) = \gamma(1 \# x)} \}.$$

If $\phi \in r\mathrm{Aut}(S)$ then $\exists\, \alpha \in \Bbbk^\times$, $I \subset \mathbb{N}$ and a finite set of odd #s $\beta_i \in \Bbbk$ s.t.

$$\begin{aligned} \phi(u \# 1) &= \alpha(u \# 1) \\ \phi(v \# 1) &= \alpha(\gamma^{-1}(v \# 1) + \sum_{i \in I} \beta_i u^i \# x) \end{aligned}$$ (even type)

or

$$\begin{aligned} \phi(u \# 1) &= \alpha(u \# g - 2v \# gx) \\ \phi(v \# 1) &= \alpha(\gamma^{-1}(v \# g) + \sum_{i \in I} \beta_i u^i \# gx) \end{aligned}$$ (odd type)

② $\mathcal{U}(A \# H)$ is the complement of the zero locus of (u^n) in $\mathrm{MaxSpec}(\Bbbk[u^n, v^n])$.