Singularity categories of some noncommutative deformations

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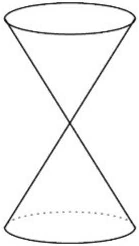
Commutative singularities

- Work over $\mathbb{k} = \mathbb{C}$.
- Commutative ring $R \rightsquigarrow$ Geometric object Spec R.
- In particular, we can study the singularities of Spec R.
- e.g. $R = \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$

Spec $R =$ 

- This is an example of an $\mathbb{A}_1$ singularity.

Noncommutative singularities

- What about if R is noncommutative?
- One issue: $\operatorname{Spec} R = \{\text{prime ideals of } R\}$ is often too small.
- "Geometric properties of $\operatorname{Spec} R$" $\leftrightarrow$ "Algebraic properties of R"
- $\rightsquigarrow$ If R is noncommutative, we say it has a geometric property if it has the corresponding algebraic property.
- Commutative fact: $\operatorname{Spec} R$ is smooth $\Leftrightarrow$ $\operatorname{gl.dim} R < \infty$.

Definition

A (possibly noncommutative) ring R is singular (resp. smooth or nonsingular) if its global dimension is infinite (resp. finite).

- We would also like the be able to describe the singularities of R.

Singularity categories

- Technical standing assumption: R is Gorenstein i.e. noetherian and $\operatorname{i.dim} R_R = \operatorname{i.dim} {}_R R < \infty$.
- In '86, Buchweitz defined the singularity category of a noetherian ring R to be

$$\mathscr{D}_{\mathrm{sg}}(R) := \frac{\mathscr{D}^{\mathrm{b}}(R)}{\operatorname{Perf} R}.$$

  This is a triangulated category (with translation Σ).

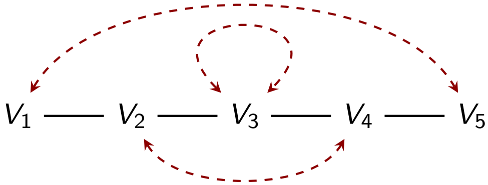
Lemma

$\mathscr{D}_{\mathrm{sg}}(R)$ is trivial $\Leftrightarrow$ R is smooth.

- Observation: don't need R to be commutative.
- We can compare the singularities of two rings by comparing their singularity categories.
- "The bigger $\mathscr{D}_{\mathrm{sg}}(R)$, the more singular R is."

Example: Kleinian singularities

- Family of surface singularities with coordinate rings $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(f)$, where G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$.
- Parametrised by simply laced Dynkin diagrams Q. Write R_Q for the corresponding coordinate ring.
- Properties of $\mathscr{D}_{\mathrm{sg}}(R_Q)$:
  - Krull-Schmidt category;
  - $\{$indecomposable objects$\} \leftrightarrow \{$vertices of $Q\}$;
  - Σ induces a graph automorphism of Q.
- e.g. $R_{\mathbb{A}_5}$

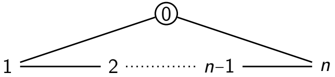


Deformations of Kleinian singularities

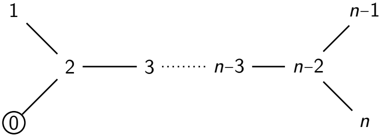
- In '98, Crawley-Boevey and Holland introduced a family of deformations $\mathcal{O}^{\lambda}(\widetilde{Q})$ of the $R_Q = \mathbb{C}[u, v]^G$.
- Require two pieces of data:
  - an extended Dynkin graph $\widetilde{Q}$;

Deformations of Kleinian singularities

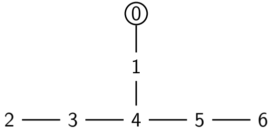
$\widetilde{\mathbb{A}}_n$



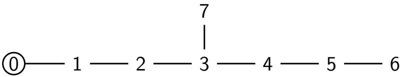
$\widetilde{\mathbb{D}}_n,\ n \geqslant 4$



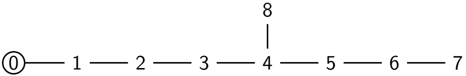
$\widetilde{\mathbb{E}}_6$



$\widetilde{\mathbb{E}}_7$



$\widetilde{\mathbb{E}}_8$



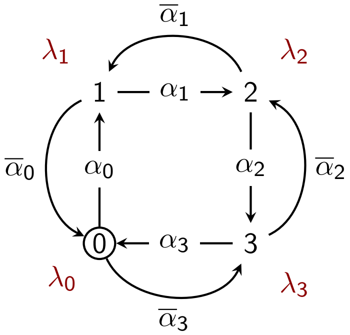
Deformations of Kleinian singularities

- In '98, Crawley-Boevey and Holland introduced a family of deformations $\mathcal{O}^{\lambda}(\widetilde{Q})$ of the $R_Q = \mathbb{C}[u, v]^G$.
- Require two pieces of data:
  - an extended Dynkin graph $\widetilde{Q}$;
  - a weight $\lambda \in \mathbb{C}^{\widetilde{Q}_0}$ (a complex number for each vertex of $\widetilde{Q}$).
- If we input a type $\widetilde{Q}$ graph, we are deforming R_Q.
- We need another definition before we can define the deformations.

Deformed preprojective algebras

- Let Q be a quiver without loops.
- Form the double $\overline{Q}$ of Q by adding a reverse arrow $\overline{\alpha} : j \to i$ for each arrow $\alpha : i \to j$.
- Choose a weight λ.
- Then the deformed preprojective algebra is $\Pi^{\lambda}(Q) = \mathbb{C}\overline{Q}/I$, where I is the two-sided ideal with generators

$$\sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \alpha\overline{\alpha} - \sum_{\substack{\alpha \in Q_1 \\ h(\alpha)=i}} \overline{\alpha}\alpha = \lambda_i e_i$$



Deformations of Kleinian singularities

- Recall that $\mathcal{O}^\lambda(\widetilde{Q})$ depends on the data of an extended Dynkin graph $\widetilde{Q}$ and a weight $\lambda \in \mathbb{C}^{\widetilde{Q}_0}$.
- Choose any orientation for the edges of $\widetilde{Q}$ to get a quiver $\widetilde{Q}$. Then

$$\mathcal{O}^\lambda(\widetilde{Q}) := e_0 \Pi^\lambda(\widetilde{Q}) e_0.$$

- Easy to write down a presentation in type $\mathbb{A}$:

$$\frac{\mathbb{C}\langle x, y, z\rangle}{\left\langle \begin{array}{ll} xz = (z + \sum_{i=0}^{n} \lambda_i)x, & xy = \prod_{i=0}^{n} \left(z + \sum_{j=1}^{i} \lambda_j \right) \\ yz = (z - \sum_{i=0}^{n} \lambda_i)y, & yx = \prod_{i=0}^{n} \left(z - \sum_{j=1}^{i} \lambda_j \right) \end{array} \right\rangle},$$

i.e., they're examples of generalised Weyl algebras.

Some properties of $\mathcal{O}^\lambda(\widetilde{Q})$

- Why are these deformations?

Lemma (Crawley-Boevey – Holland)

(1) $\mathcal{O}^{\mathbf{0}}(\widetilde{Q}) \cong R_Q$.

(2) *There exists a filtration of* $\mathcal{O}^\lambda(\widetilde{Q})$ *such that* $\operatorname{gr} \mathcal{O}^\lambda(\widetilde{Q}) \cong R_Q$.

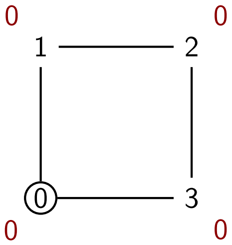
- It is also easy to detect when $\mathcal{O}^\lambda(\widetilde{Q})$ is commutative:

Lemma (Crawley-Boevey – Holland)

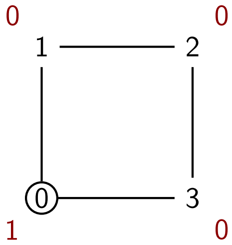
$\mathcal{O}^\lambda(\widetilde{\mathbb{A}}_n)$ *is commutative iff* $\sum_{i=0}^n \lambda_i = 0$ *(there are similar conditions for the other types).*

Example

- Consider $\mathcal{O}^{\lambda}(\widetilde{Q})$ for the following data:



Commutative



Noncommutative

The problem

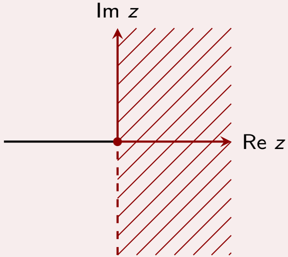
Goal

Determine $\mathscr{D}_{\text{sg}}(\mathcal{O}^\lambda(\widetilde{Q}))$.

- This is difficult for arbitrary λ. We can simplify matters, but first:

Definition

Call a weight quasi-dominant if λ_i "lies in the right-half of the complex plane" for all $i \geqslant 1$.

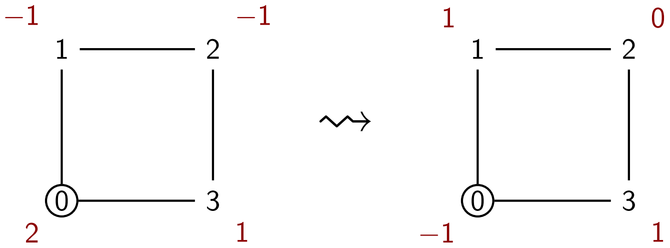


Simplifying the problem

Lemma (Boddington – Levy)

Given a weight λ for $\widetilde{Q}$, there exists a quasi-dominant weight λ' such that $\mathcal{O}^{\lambda}(\widetilde{Q}) \cong \mathcal{O}^{\lambda'}(\widetilde{Q})$.

- Henceforth, assume all weights are quasi-dominant.
- Aside: there is an algorithm to find λ'.



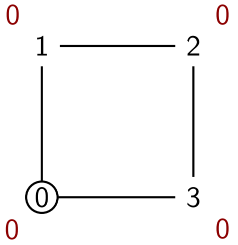
Detecting smoothness

- This makes it easy to detect smoothness:

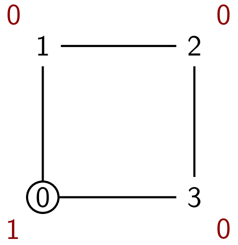
Lemma (Crawley-Boevey – Holland)

$\mathcal{O}^\lambda(\widetilde{Q})$ *is singular iff* $\lambda_i = 0$ *for some* $i \neq 0$.

- For example:



Commutative and singular



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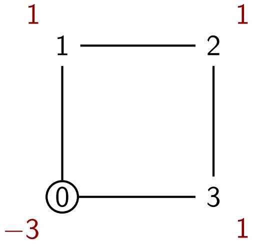
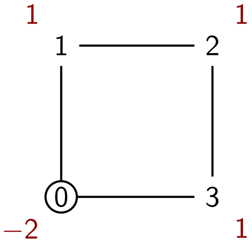


Commutative and smooth



Noncommutative and smooth

The main result

Theorem (C., 2016)

Let $\widetilde{Q}$ be an extended Dynkin graph, and let λ be a quasi-dominant weight for $\widetilde{Q}$. Write Q_λ for the full subgraph of $\widetilde{Q}$ obtained by removing

- *vertex 0, and*
- *each vertex $i \geqslant 1$ with $\lambda_i \neq 0$.*

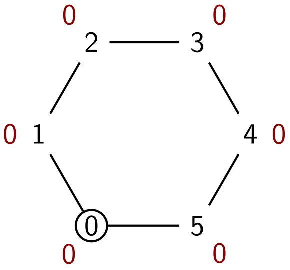
Then $Q_\lambda = \bigsqcup_{i=1}^{r} Q_i$ is a disjoint union of Dynkin graphs, and there is a triangle equivalence

$$\mathscr{D}_{\mathsf{sg}}(\mathcal{O}^{\lambda}(\widetilde{Q})) \simeq \bigoplus_{i=1}^{r} \mathscr{D}_{\mathsf{sg}}(R_{Q_i}).$$

Example: $\widetilde{Q} = \widetilde{\mathbb{A}}_5$

Theorem (C., 2016)

Let $\widetilde{Q} \in \{\widetilde{\mathbb{A}}, \widetilde{\mathbb{D}}, \widetilde{\mathbb{E}}\}$ and λ be quasi-dominant. Let Q_λ be the full subgraph of $\widetilde{Q}$ obtained by removing vertex 0 and each vertex $i \geqslant 1$ with $\lambda_i \neq 0$. Then $Q_\lambda = \bigsqcup_{i=1}^r Q_i$, ($Q_i$ Dynkin), and $\mathscr{D}_{\mathrm{sg}}(\mathcal{O}^\lambda(\widetilde{Q})) \simeq \bigoplus_{i=1}^r \mathscr{D}_{\mathrm{sg}}(R_{Q_i})$.



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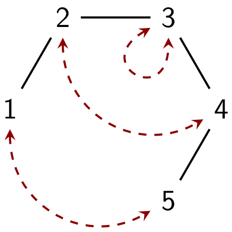

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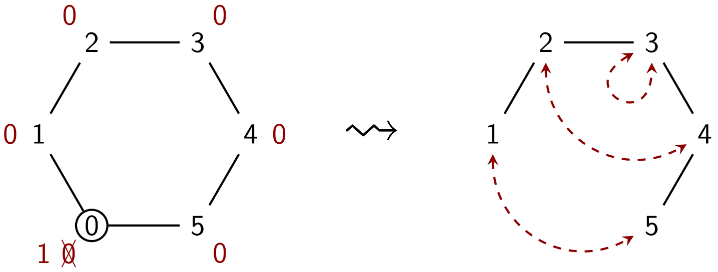

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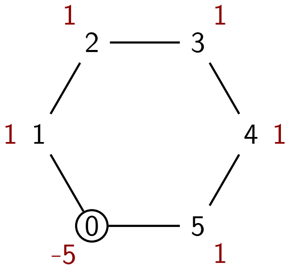

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$$\mathscr{D}_{\mathrm{sg}}(\mathcal{O}^\lambda(\widetilde{Q})) \qquad \simeq \qquad \{0\}$$

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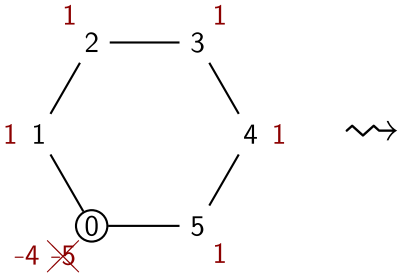

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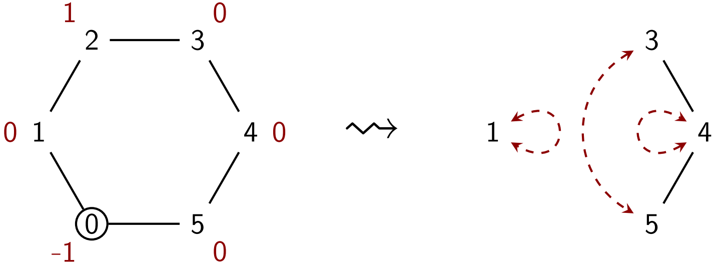

$$\mathscr{D}_{\text{sg}}(\mathcal{O}^\lambda(\widetilde{Q})) \simeq \mathscr{D}_{\text{sg}}(R_{\mathbb{A}_1}) \oplus \mathscr{D}_{\text{sg}}(R_{\mathbb{A}_3})$$

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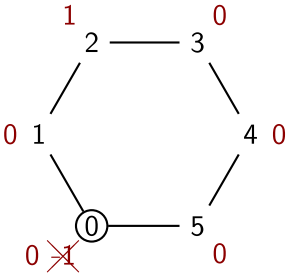

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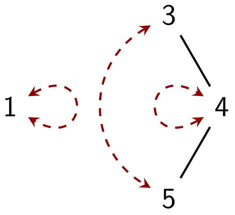

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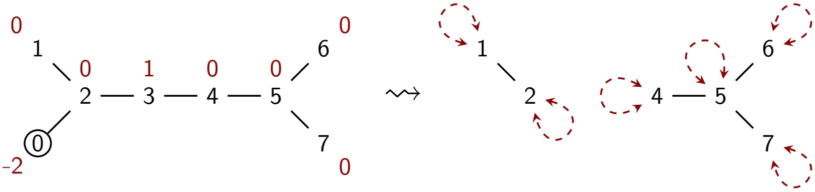

$\mathscr{D}_{\mathrm{sg}}(\mathcal{O}^\lambda(\widetilde{Q})) \simeq \mathscr{D}_{\mathrm{sg}}(R_{\mathbb{A}_2}) \oplus \mathscr{D}_{\mathrm{sg}}(R_{\mathbb{D}_4})$

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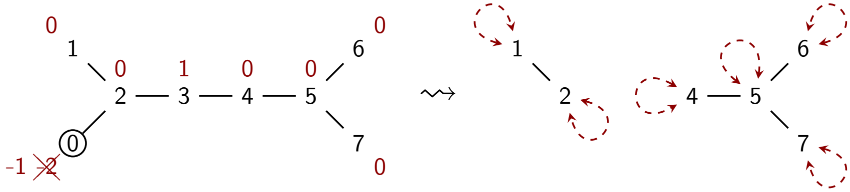

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Some remarks

- Intuition: deforming a singularity should make it no more singular. This is true for deformations of Kleinian singularities.
- If λ is quasi-dominant, $\mathcal{O}^{\lambda}(\widetilde{Q})$ is commutative, and $\mu = (\lambda_0 + 1, \lambda_1, \dots, \lambda_n)$, then we think of $\mathcal{O}^{\mu}(\widetilde{Q})$ as a noncommutative analogue of $\mathcal{O}^{\lambda}(\widetilde{Q})$.
  - They have the same singularity categories.
  - If $\lambda = \mathbf{0}$ and μ is as above, then there is a noncommutative version of the geometric McKay correspondence.

Crawley-Boevey – Holland's deformations done right

- Crawley-Boevey – Holland's original paper introduced the deformations differently. Here's what they actually did:
- Let $G \underset{\text{fin}}{\leqslant} \mathrm{SL}(2, \mathbb{C})$ with associated extended Dynkin graph $\widetilde{Q}$ and let $S = \mathbb{C}[u, v] \# G$.
- Crawley-Boevey – Holland showed that one can deform S to get an algebra $\mathcal{S}^\lambda$ and that
  - $\mathcal{S}^\lambda \sim \Pi^\lambda(\widetilde{Q})$; and
  - if $e = \frac{1}{|G|} \sum_{g \in G} g$, then $e\mathcal{S}^\lambda e \cong \mathcal{O}^\lambda(\widetilde{Q})$.
- Can we replace $\mathbb{C}[u, v]$ and G with sensible alternatives and get similar results?

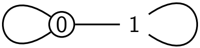
A noncommutative generalisation of CBH's work

- Chan, Kirkman, Walton & Zhang recently classified all pairs (A, H) where:
  - A is an AS-regular algebra of global dimension 2; and
  - H is a semisimple Hopf algebra acting inner faithfully on A with trivial homological determinant.
- $\rightsquigarrow$ These actions are like the actions of finite subgroups of $\mathrm{SL}(n, \mathbb{C})$ on $\mathbb{C}[x_1, \dots, x_n]$.
- CKWZ have shown that analogues of results in the Auslander-McKay correspondence for finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ hold for the pairs (A, H).
- I'll restrict attention to the case where $H = \mathbb{C}G$ for some group G. How much of CBH's work generalises?

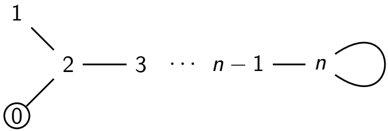
The pairs (A, G)

Case	A	G	$\widetilde{Q}$
(0)	$\mathbb{C}[u,v]$	$G \underset{\text{fin}}{\leqslant} \mathrm{SL}_2(\mathbb{C})$	$\widetilde{\mathbb{A}}$-$\widetilde{\mathbb{D}}$-$\widetilde{\mathbb{E}}$
(i)	$\mathbb{C}_q[u,v]$	C_{n+1}	$\widetilde{\mathbb{A}}_n$
(ii)	$\mathbb{C}_{-1}[u,v]$	C_2	$\widetilde{\mathbb{L}}_1$
(iii)	$\mathbb{C}_{-1}[u,v]$	D_n	$\begin{cases} \widetilde{\mathbb{D}}_{\frac{n+4}{2}} & n \text{ even} \\ \widetilde{\mathbb{DL}}_{\frac{n+1}{2}} & n \text{ odd} \end{cases}$
(iv)	$\mathbb{C}_J[u,v]$	C_2	$\widetilde{\mathbb{A}}_1$

$\widetilde{\mathbb{L}}_1$



$\widetilde{\mathbb{DL}}_n$



Deformations of $A \# G$ and A^G

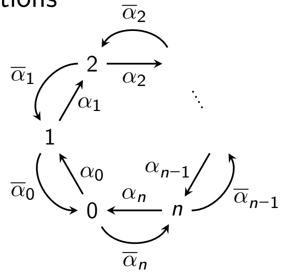
- Fact: $e(A \# G)e \cong A^G$.
- One can deform the algebras $A \# G$ and A^G in the same way as CBH did to get algebras $(A \# G)^\lambda$ and $e(A \# G)^\lambda e$, where $\lambda \in \mathbb{C}^{\widetilde{Q}_0}$.
- These deformations have nice properties:

Proposition (C.)

- $(A \# G)^\lambda$ is a prime, noetherian, finitely generated $\mathbb{C}$-algebra. It is Auslander-regular of global dimension $\leqslant 2$, and Cohen-Macaulay of GK dimension 2.
- $e(A \# G)^\lambda e$ is a finitely generated $\mathbb{C}$-algebra which is a noetherian domain. It is Auslander-Gorenstein, and Cohen-Macaulay of GK dimension 2.

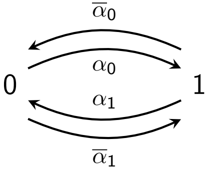
Deformed "preprojective algebras"

- Fix $q \in \mathbb{C}^\times$. Define the quantum deformed preprojective algebra $\Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ as the path algebra with relations



$$\alpha_i\overline{\alpha}_i - q\alpha_{i-1}\overline{\alpha}_{i-1} = \lambda_i e_i.$$

- Define Δ^λ as the path algebra with relations



$$\alpha_0\overline{\alpha}_0 - \overline{\alpha}_1\alpha_1 - \alpha_0\alpha_1 = \lambda_0 e_0$$
$$\alpha_1\overline{\alpha}_1 - \overline{\alpha}_0\alpha_0 - \alpha_1\alpha_0 = \lambda_1 e_1.$$

Morita equivalences and isomorphisms between deformations

- Let $G \underset{\text{fin}}{\leqslant} \mathrm{SL}(2, \mathbb{C})$ with associated extended Dynkin graph $\widetilde{Q}$. Then Crawley-Boevey – Holland's results can be written as
  - $(\mathbb{C}[u,v] \# G)^\lambda \sim \Pi^\lambda(\widetilde{Q})$; and
  - $e(\mathbb{C}[u,v] \# G)^\lambda e \cong e_0 \Pi^\lambda(\widetilde{Q}) e_0$.
- These results generalise to our new setting:

Theorem (C., 2016)

Case (i): $(\mathbb{C}_q[u,v] \# C_{n+1})^\lambda \sim \Pi_q^\lambda(\widetilde{\mathbb{A}}_n)$ and $e(\mathbb{C}_q[u,v] \# C_{n+1})^\lambda e \cong e_0 \Pi_q^\lambda(\widetilde{\mathbb{A}}_n) e_0$.
Cases (ii)-(iii): $(A \# G)^\lambda \sim \Pi^\lambda(\widetilde{Q})$ and $e(A \# G)^\lambda e \cong e_0 \Pi^\lambda(\widetilde{Q}) e_0$.
Case (iv): $(\mathbb{C}_J[u,v] \# C_2)^\lambda \sim \Delta^\lambda$ and $e(\mathbb{C}_J[u,v] \# C_2)^\lambda e \cong e_0 \Delta^\lambda e_0$.

Auslander's Theorem for the deformations

- We have the following well-known theorem:

Auslander's Theorem (1962)

Let $G \underset{\text{fin}}{\leqslant} \mathrm{GL}(n, \mathbb{C})$ be a small group acting on $S := \mathbb{C}[x_1, \dots, x_n]$. Then $\mathrm{End}_{S^G}(S) \cong S \# G$.

- Chan-Kirkman-Walton-Zhang recently proved the following:

Theorem (Chan-Kirkman-Walton-Zhang, 2016)

Let (A, G) be a pair from the earlier table. Then $\mathrm{End}_{A^G}(A) \cong A \# G$.

- A slightly stronger result can be proved using different techniques:

Theorem (C., 2017)

The deformations $(A \# G)^\lambda$ are maximal orders, and $\mathrm{End}_{e(A \# G)^\lambda e}(A) \cong (A \# G)^\lambda$.

Future questions

- What do the singularity categories of the deformations $e(A \# G)^{\lambda}e$ look like?
    - When $\lambda = \mathbf{0}$ (so $e(A \# G)^{\lambda}e \cong A^G$), I can answer this.
- How do the global dimensions of $(A \# G)^{\lambda}$ and $e(A \# G)^{\lambda}e$ vary with λ?
- How does the number of finite dimensional simple modules over $(A \# G)^{\lambda}$ and $e(A \# G)^{\lambda}e$ vary with λ?
- Is $(A \# G)^{\lambda}$ ever Morita equivalent to $e(A \# G)^{\lambda}e$?