Tracing a Path – From Walks on Graphs to Invariant Theory (and Beyond)

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University of Wisconsin-Madison

Algebra Extravaganza!
A Conference in Honor of Ellen Kirkman and Martin Lorenz
Temple University
July 25, 2017

A Walkthrough

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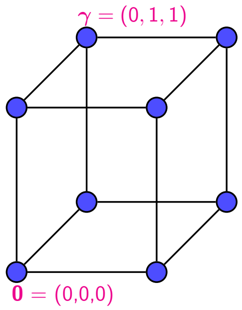

How many walks of k steps

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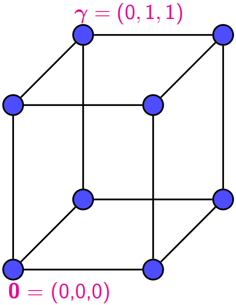

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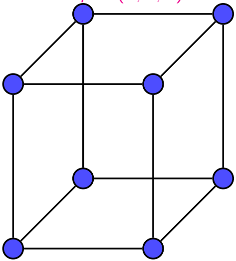

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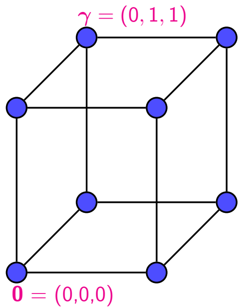

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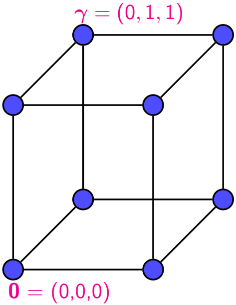

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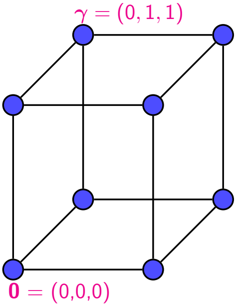

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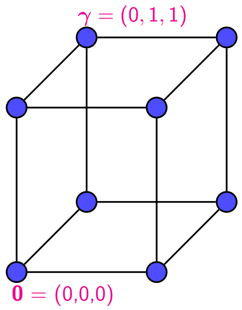

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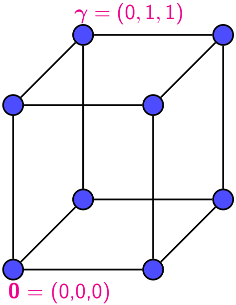

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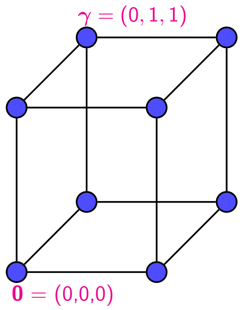

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and ways to study them using characters

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- McKay matrix: $A = (a_{\alpha,\gamma})_{\alpha,\gamma\in\Gamma}$

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$$\chi_0 \cdot \underbrace{\chi_V \cdot \chi_V \cdot \cdots \cdot \chi_V}_{\substack{k \text{ times} \\ (\text{character of } V^{\otimes k})}} = \sum_{\gamma \in \Gamma} \underbrace{(A^k)_{0,\gamma}}_{\substack{\#\text{walks } k \text{ steps } 0 \text{ to } \gamma \\ (\text{multiplicity of } S_\gamma \text{ in } V^{\otimes k})}} \chi_\gamma$$

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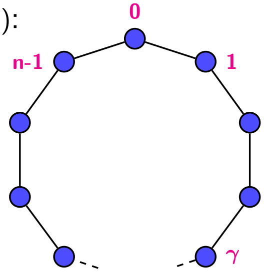

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&= n^{-1} \sum_{\beta=0}^{n-1} \omega^{\alpha\beta} \left(\omega^{-\beta} + \omega^{\beta}\right)^k\ \omega^{-\gamma\beta} \\
&= n^{-1} \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\sum_{\beta=0}^{n-1} \omega^{(\alpha-\gamma+k-2\ell)\beta} \right) \\
&= \sum_{k-2\ell \,\equiv\, \gamma-\alpha \mod n} \binom{k}{\ell}
\end{aligned}$$

since $\displaystyle \sum_{\beta=0}^{n-1} \omega^{m\beta} = \begin{cases} n & \text{if } m \equiv 0 \mod n \\ 0 & \text{otherwise.} \end{cases}$

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$$\mathrm{h}_j(t, r) := r^{-1} \sum_{b=0}^{r-1} \omega^{(1-j)b} \mathrm{e}^{\omega^b t} = \sum_{\ell=0}^{\infty} \frac{t^{r\ell+j-1}}{(r\ell + j - 1)!}$$

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$$= \left(r_1^{-1} \sum_{\beta_1=0}^{r_1-1} \omega_1^{-\gamma_1\beta_1} \mathrm{e}^{\omega_1^{\beta_1} t} \right) \cdots \left(r_n^{-1} \sum_{\beta_n=0}^{r_n-1} \omega_n^{-\gamma_n\beta_n} \mathrm{e}^{\omega_n^{\beta_n} t} \right)$$

For $\omega = \mathrm{e}^{2\pi i/r}$, consider the *generalized hyperbolic function*

$$\mathrm{h}_j(t,r) := r^{-1} \sum_{b=0}^{r-1} \omega^{(1-j)b} \mathrm{e}^{\omega^b t} = \sum_{\ell=0}^{\infty} \frac{t^{r\ell+j-1}}{(r\ell+j-1)\,!}$$

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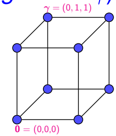

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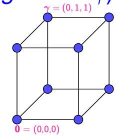

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Affine Dynkin diagrams

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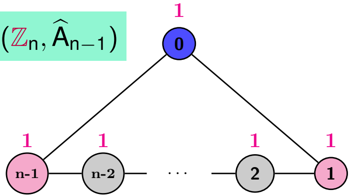

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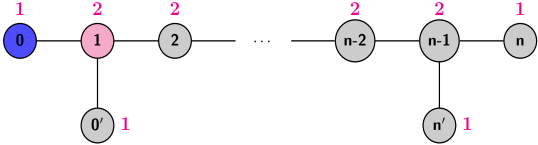
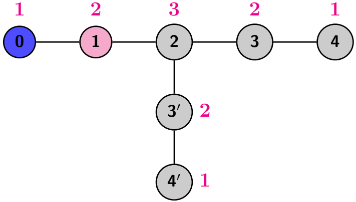
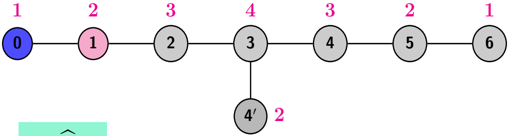


$(\mathbb{T}, \widehat{\mathsf{E}}_6)$



$(\mathbb{O}, \widehat{\mathsf{E}}_7)$



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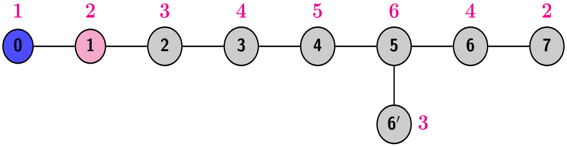

Cor. When $G \subset SU_2$ is finite and $G \ncong \mathbb{Z}_n$ for n odd, and $V = \mathbb{C}^2$, then the Poincaré series for the G-invariants $T(V)^G$ in $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ is

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- $\mathring{A}$: adjacency matrix of the corresponding finite Dynkin diagram;
- $\widehat{m}$ and $\widehat{h}$: exponents and Coxeter number of the affine root system;
- m and h: exponents and Coxeter number of the finite root system.

Cor. When $\mathsf{G} \subset \mathsf{SU}_2$ is finite and $\mathsf{G} \ncong \mathbb{Z}_n$ for n odd, and $\mathsf{V} = \mathbb{C}^2$, then the Poincaré series for the G-invariants $\mathsf{T}(\mathsf{V})^{\mathsf{G}}$ in $\mathsf{T}(\mathsf{V}) = \bigoplus_{k \geq 0} \mathsf{V}^{\otimes k}$ is

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Springer's proof uses Molien's formula

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A dynamical view of group walking – Chip firing

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$$\underline{\mathbf{d}} := \begin{pmatrix} d_0 = 1 \\ d_1 \\ \vdots \\ d_\ell \end{pmatrix} = \begin{pmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(e) \end{pmatrix}$$ is a null vector for $\widehat{C}$

joint with C. Klivans and V. Reiner (2016)

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(i.e. chip firing stabilizes) for all such G, V.

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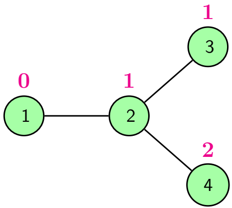

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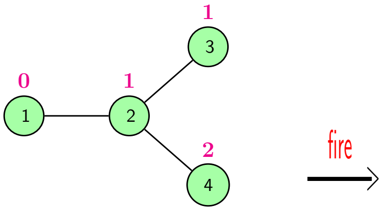

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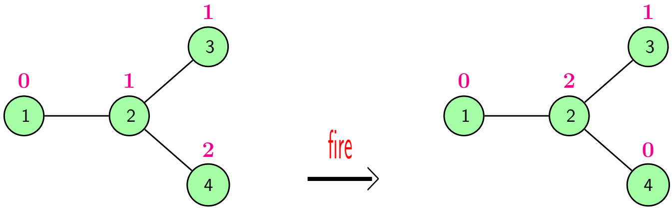

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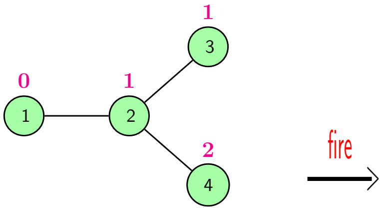

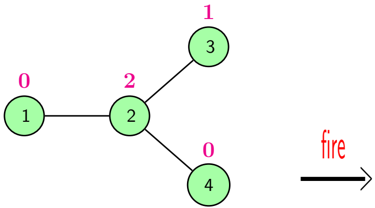

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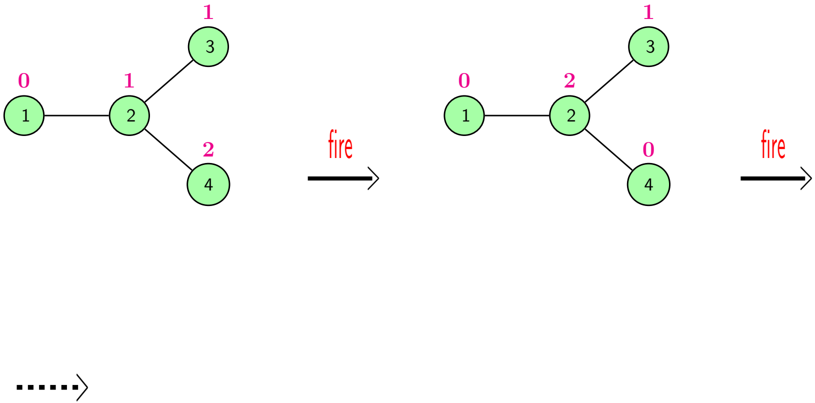

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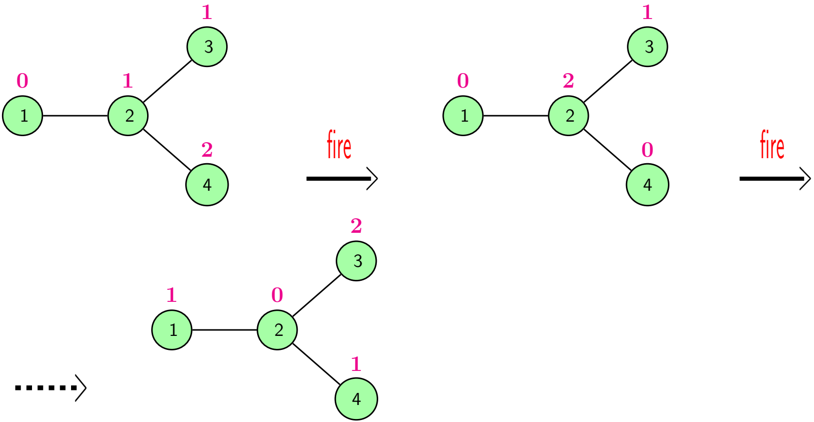

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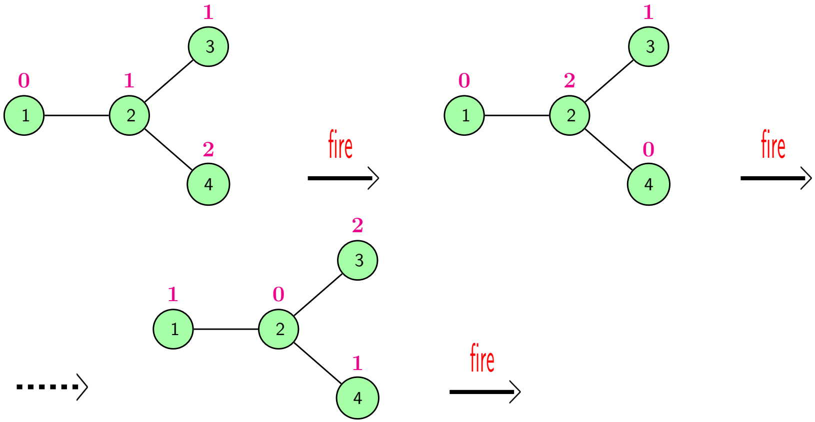

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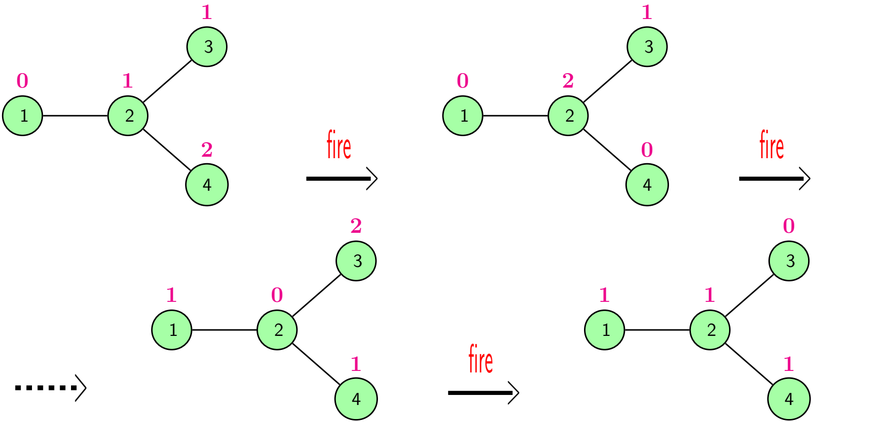
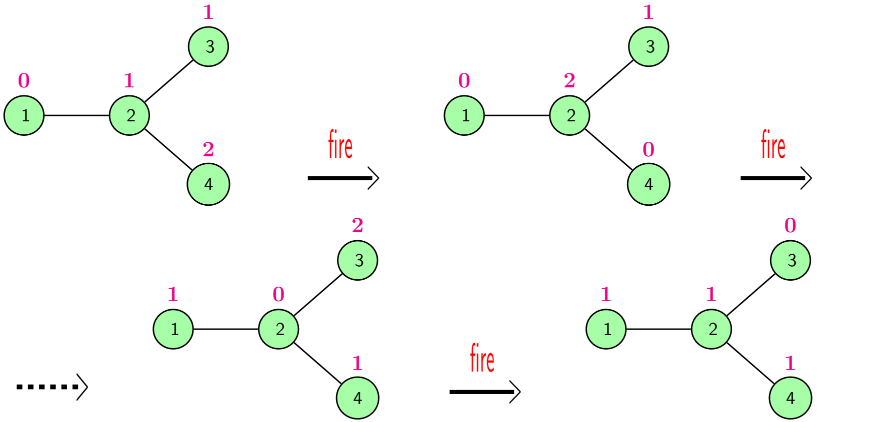


Example of an avalanche-finite matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \qquad C = \widehat{C} \setminus \{\mathrm{Row}_0, \mathrm{Column}_0\} \text{ for } \widehat{C} \text{ of type } \widehat{D}_4$$

(C is the Cartan matrix of finite type D_4)



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A configuration $\underline{a}$ is *recurrent if there is a burning configuration* $\underline{b}$ *so*

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Thm *[B.-Klivans-Reiner '16]* *If* C *an honest Cartan matrix, then*

- *The burning configurations are the nonzero elements of the root lattice in the fundamental chamber.*
- *The recurrent configurations are* ϱ *(half-sum of the positive roots) and* $\varrho - \lambda_i$*, for* λ_i *a minuscule weight.*

⋮

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(e) For V, let $A_{i,j} := [S_j \otimes V, S_i]$, and set $A = (A_{i,j})$.

(f) Let $\widehat{C} = d\mathrm{I} - A$.

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- And there are a lot of other beautiful results related to Brauer characters, chip firing, Cartan matrices (in the modular group theory sense), critical groups, etc.

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When $0 \leq r \leq p-2$, the module P_r has the following structure:

$$\mathrm{P}_r : \qquad \begin{array}{ccccc} & & \mathrm{S}_r & & \\ & \swarrow & & \searrow & \\ \mathrm{S}_{p-2-r} & & & & \mathrm{S}_{p-2-r} \\ & \searrow & & \swarrow & \\ & & \mathrm{S}_r & & \end{array}$$

$\mathrm{P}_{p-1} = \mathrm{S}_{p-1}$, the Steinberg module.

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Tensor product rules (B.-Osborn '82):

$$S_r \otimes S_1 = S_{r+1} \oplus S_{r-1} \quad \text{for } 0 \leq r \leq p-2$$

$$S_{p-1} \otimes S_1 = P_{p-2}, \qquad \text{so}$$

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Remark There is an analog of this for the "restricted" quantum group $u_\xi(\mathfrak{g})$ for an ℓth root of unity ξ, ℓ odd, and $K = \mathbb{C}$. Calculations are basically the same - just replace p by ℓ. (See Chari-Premet '94)

Thanks

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